VIBRATIONS OF SERIES OF BEAMS CONNECTED BY FLEXIBLE NONLINEAR LAYERS WITH APPLICATION TO CARBON NANOTUBES

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VIBRATIONS OF SERIES OF BEAMS CONNECTED BY FLEXIBLE NONLINEAR LAYERS WITH APPLICATION TO CARBON NANOTUBES

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Thesis

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ABSTRACT

Recent modeling of multi-walled carbon nanotubes by a series of elastically connected Euler-Bernoulli beams has led to predictions of frequencies and mode shapes for non-coaxial modes of vibration. Whereas previous work assumes the van der Waals forces between atoms are modeled by elastic layers, this work assumes such forces are modeled by a flexible layer with quadratic nonlinearities. The nonlinear terms are derived from an expansion of the Lennard-Jones potential function for small changes from an equilibrium position.

The current work assumes a model for the free vibrations of multi-walled carbon nanotubes of the free vibrations of a series of Euler-Bernoulli beams connected by flexible layers with quadratic nonlinearities. The method of multiple scales is used to derive uniformly valid asymptotic expansions for the free responses of the beams. Secular terms arising from internal resonances are identified and eliminated. The resulting equations for amplitude and phase are solved numerically. An example is presented.

The results can be used to refine predictions of free responses of carbon nanotubes. The work can be extended to identify the effects of combination resonances resulting from harmonic excitation.
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CHAPTER I

INTRODUCTION

1.1 Motivation

With the encroachment of science and technology, the era of nanotechnology has evolved. Nanotechnology is the creation of functional materials, devices and systems through control of matter on the nanometer length scale (1-100 nanometers), and exploitation of novel phenomena and properties (physical, chemical, biological, mechanical, electrical...) at that length scale. The most distinct characteristic of nanotechnology is that the properties of nanomaterials are size dependent.

A nanotube is a nanometer scale wire-like structure that is most often composed of carbon. Nanotubes are long, slender fullerness, cylindrical, with at least one end typically capped with a hemisphere of buckyball structure. Fullerenes, or Buckminsterfullerenes, are named after Buckminster Fuller, the architect and designer of the geodesic dome, and are sometimes called bucky balls. The names derive from the basic shape that defines fullerenes (7) an elongated sphere of carbon atoms formed by interconnecting six-member rings, and twelve isolated five-member rings forming hexagonal and pentagonal
faces. Among many nano structured materials, carbon nanotubes attracted wide attention and stimulated extensive experimental and theoretical studies.

Carbon nanotubes are a recently discovered allotrope of carbon, with seemingly endless list of abilities. In 1991, Iijima discovered closed tabular structures consisting of nested cylindrical graphitic layers capped by Fullerene like ends with a hollow internal cavity. Since then the experimental and theoretical studies have grown exponentially due to their novel properties that make them potentially useful in a wide variety of applications in nanotechnology, electronics, and other fields of material science. Since their discovery 15 years ago, carbon nanotubes have captured the imagination of scientists and lay people alike. These structures, so minuscule they cannot be seen, are stronger than diamonds. They are formed from organic material but act as metals or semi-conductors. They offer great potential in electronics, lasers and medicine.

Nanotubes are sheets of carbon atoms connected in a honeycomb-like pattern that are rolled into tiny cylinders one nanometer in diameter. No other element in the periodic table bonds to itself in an extended network with the strength of the carbon-carbon bond. The delocalised pi-electron donated by each atom is free to move about the entire structure, rather than stay home with its donor atom, giving rise to the first molecule with metallic-type electrical conductivity. One nanometer is one one-billionth of a meter, or one 10,000th the thickness of a human hair. The properties of the tube depend on how the cylinder is rolled, just as the properties of chemical elements depend on their weight. Carbon nanotubes that Iijima (8) observed were so called multi walled tubes-each contained a one another like Russian dolls. In diamond the carbon atoms link into four
sided tetrahedral but in nanotubes the atoms arrange themselves in hexagonal rings like chicken wire.

Carbon nanotubes consist of either one cylindrical graphite sheet (single wall nanotube-SWNT) or of several nested cylinders with an interlayer spacing of 0.34-0.36nm (multiwall nanotubes-MWNT). SWNT are usually closed at both ends by Fullerene like half spheres that contain both pentagons and hexagons. The length of SWNTs and MWNTs are usually well over 1 mm and diameters range from 1 nm (for SWNT) TO 50 nm (for MWNT), leading to a very high aspect ratio (length/diameter). It
is an ongoing and challenging process to identify an effective and an efficient computational method to evaluate the properties of nanomaterials.

1.2 Synthesis of Carbon nanotubes

There are several methods by which carbon nanotubes can be produced, namely, laser ablation, chemical vapor deposition method and most importantly arc discharge method, produces the best quality nanotubes, which involves passing a current of about 50 amps between two graphite electrodes in an atmosphere of helium. This causes the graphite to vaporise, some of it condensing on the walls of the reaction vessel and some of it on the cathode. It is the deposit on the cathode which contains the carbon nanotubes. Single-walled nanotubes are produced when Co and Ni or some other metal is added to the anode. It has been known since the 1950s, if not earlier, that carbon nanotubes can also be made by passing a carbon-containing gas, such as a hydrocarbon, over a catalyst. The catalyst consists of nano-sized particles of metal, usually Fe, Co or Ni. These particles catalyse the breakdown of the gaseous molecules into carbon, and a tube then begins to grow with a metal particle at the tip.

It was shown in 1996 that single-walled nanotubes can also be produced catalytically. The perfection of carbon nanotubes produced in this way has generally been poorer than those made by arc-evaporation, but great improvements in the technique have been made in recent years. The big advantage of catalytic synthesis over arc-evaporation is that it can be scaled up for volume production. The third important method for making carbon nanotubes involves using a powerful laser to vaporise a metal-graphite target.
This can be used to produce single-walled tubes with high yield. Among the above mentioned methods, arc discharge method and chemical vapor deposition method produces both single and double walled nanotubes, where as laser ablation method produces only single walled nanotubes.

1.3 Properties of carbon nanotubes

The strength of the \( \text{sp}^2 \) carbon-carbon bonds gives carbon nanotubes amazing mechanical properties. The mechanical and electrical properties of carbon nanotubes have been the subject of numerous studies ever since the discovery of nanotubes, largely because of the wide recognition of carbon nanotubes as important nanoscale systems and because of various envisioned novel applications. Carbon nanotubes which are formed of seamless cylindrical graphene layers represent the ideal carbon fiber and should presumably exhibit the best mechanical properties. This feature is probably the most promising for applications of nanotubes given the importance of extremely strong light-weight composites. The mechanical stiffness and strength of carbon nanotubes are expected to be very high.

It is remarkably difficult to directly measure the mechanical properties of single nanotubes due to their small diameters and because they tend to form bundles. The stiffness of carbon nanotubes was recently measured by a thermal vibration technique and young’s modulus was reported to be in the range of 1-5 TPa (10). The Young's modulus of the best nanotubes can be as high as 1000 GPa which is approximately 5x higher than steel. The tensile strength, or breaking strain of nanotubes can be up to 63 GPa, around
50x higher than steel. These properties, coupled with the lightness of carbon nanotubes, give them great potential in applications such as aerospace.

The electronic properties of carbon nanotubes are also extraordinary. The unique physical and chemical properties of CNTs, such as structural rigidity and flexibility continue to generate considerable interest. Additionally, CNTs are extremely strong, about 100 time’s stronger (stress resistant) than steel at one-sixth the weight. CNTs can also act as either conductors or semiconductors depending on their chirality, possess an intrinsic superconductivity, are ideal thermal conductors, and can also behave as field emitters. Currently, amazing properties of this material enable to foresee more than 60 potential applications of carbon nanotubes, in various fields like electronic, aerospace, biomedical etc., Carbon nanotubes enable to strongly improve material characteristics, or to think to revolutionary applications related to the tubes properties (field emission, molecules storage...). The properties of carbon nanotubes such as the elastic modulus, poisson’s ratio and bulk modulus exhibit strong dependence on their helicity and diameter, stiffness, flexibility and the strength of carbon nanotubes and thus motivated researchers to study the fundamentals of this material as well as to explore their applications in different fields.

1.3.1 Mechanical Properties

1. 100 times stronger than steel (stress) and 6 times lighter.
2. High flexibility of carbon nanotubes, unlike carbon fibers.
3. Expansion by charge injection.
4. High Young’s modulus and tensile strength.

5. Large current density and high aspect ratio.

Table 1.1 Young's modulus, tensile strength and density of carbon nanotubes compared with some other materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s Modulus(GPa)</th>
<th>Tensile Strength(GPa)</th>
<th>Density(g/cm³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SWNT</td>
<td>1054</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>MWNT</td>
<td>1200</td>
<td>150</td>
<td>2.6</td>
</tr>
<tr>
<td>Steel</td>
<td>208</td>
<td>0.4</td>
<td>7.8</td>
</tr>
<tr>
<td>Epoxy</td>
<td>3.5</td>
<td>0.005</td>
<td>1.25</td>
</tr>
<tr>
<td>Wood</td>
<td>16</td>
<td>0.008</td>
<td>0.6</td>
</tr>
</tbody>
</table>

1.3.2 Electrical Properties

1. Metallic or semiconductor behavior, according to chirality's (way a graphene sheet is rolled on itself).

2. As conductive as copper.

3. Easy turnable field emission.

1.3.3 Thermal Properties

1. Carbon nanotubes are stable at high temperature.

2. Carbon nanotubes are stable in air environment.

3. Carbon nanotubes are very strong against strong acid and high temperature because of their perfect conjugated system. Acid and heat are often applied to purify carbon nanotubes.
1.3.4 Physico-chemical properties

1. High specific area (several hundreds of sqm. per gram).
2. Cavities enabling molecules storage inside the carbon nanotube
3. Chemical treatment on carbon nanotubes, enabling to fix other molecules to carbon nanotubes
4. High thermal resistance (up to 1500 °C under vacuum)

1.3.5 Field emission properties

CNTs have excellent materials properties which make them have attractive field emission characteristics

1. Large aspect ratio (>1000).
2. Atomically sharp tips.
3. High temperature and chemical stability.
4. High electrical and thermal conductivity.

1.4 Structure of carbon nanotubes

The bonding in carbon nanotubes is sp², with each atom joined to three neighbours, as in graphite. The tubes can therefore be considered as rolled-up graphene sheets (graphene is an individual graphite layer) (12). There are three distinct ways in which a graphene sheet can be rolled into a tube. The first two of these, known as “armchair” (top left) and “zig-zag” (middle left) have a high degree of symmetry. The terms "armchair" and "zig-zag" refer to the arrangement of hexagons around the circumference. The third class of
tube, which in practice is the most common, is known as chiral, meaning that it can exist in two mirror-related forms. The chiral vector, also known as the roll-up vector, can be described by equation \( \mathbf{r} = n\mathbf{a} + m\mathbf{b} \), where the integers \((n, m)\) are the number of steps along the zigzag carbon bonds of the hexagonal lattice, \(\mathbf{a}_1\) and \(\mathbf{a}_2\) are unit vectors (1). The chiral angle determines the amount of twist in the tube.

The chiral angles are 0° and 30° for the two limiting cases which are referred to as zigzag and armchair, respectively. In terms of the roll-up vector, the zigzag nanotube is denoted by \((n, 0)\) and the armchair nanotube as \((n, n)\). The roll-up vector of the nanotube also defines the nanotube diameter.

Fig 1.3 Definition of roll-up vector as linear combinations of base vectors \(a\) and \(b\).
The physical properties of carbon nanotubes are sensitive to their diameter, length and chirality. In particular, tube chirality is known to have strong influence on the electronic properties of carbon nanotubes. Nanotubes can be either metallic or semi-conducting, depending on tube chirality. Single walled carbon nanotube can undergo enormous bending deformation and recover apparently elastically. A multi walled carbon nanotube is composed of concentric graphitic cylinders and the spacing between each graphite layer is 0.34 nanometers. Both single walled and multi walled carbon nanotubes are being used as key components in the production of high strength composites and advanced sensors, electronic and optical devices, catalysts, batteries and fuel cells.

1.5 Applications of carbon nanotubes

1. Reinforcement for composite materials, space elevator.
2. Actuators.
5. Electronical nanocomponents (diodes, transistors...).
6. Electron gun, field emission display, AFM tip.
9. Electronical nanocomponents, material protection.
1.6 Vibrations

Vibrations have been long known for its capacity for disturbance, discomfort, damage and destruction. Due to practical importance, vibration analysis becomes essential, especially when the vibratory response needs to be accurately determined in order to avoid any resonances caused by the internal or external forces. The fundamental elastic systems such as the elastic beams have received a lot of attention for the development of some control theory for partial differential equations. The familiar Euler-Bernoulli beam theory has been subjected to many investigations. Because the principle of linear superposition is not valid for nonlinear systems, nonlinear systems are much more difficult to analyze than the linear systems. Solutions to nonlinear problems exist only for a few vibration problems.

For small oscillations the response of a deformable body can be adequately described by nonlinear equations and boundary conditions. However as the amplitude of oscillation increases, nonlinear effects come in to play. The source of nonlinearity may be geometric, inertial, or material in nature. The geometric nonlinearity may be caused by nonlinear stretching or large curvatures. Most of the existing studies deal with nonlinear stretching.

Non linear inertial effects are caused by the presence of concentrated or disturbed masses. Material nonlinearity occurs whenever the stresses are nonlinear functions of the strains. For a system having quadratic nonlinearities, internal and combination
resonances might exist. For a system having quadratic nonlinearities, Nayfeh, Mook and Marshal (2) found that under some conditions there exists no steady state motion inspite of the presence of damping. In this case energy is continuously exchanged between two modes without being attenuated.

We restrict our attention here to uniform systems with simple boundary conditions whose natural frequencies are obtained analytically. Then the method of multiple scales is directly applied to the partial differential equations and the boundary conditions of the nonlinear uniform beam. The nonlinear mode shapes and natural frequencies for a class of n-dimensional system with weak quadratic nonlinearities are constructed. Internal resonances occur for different natural frequencies. The nonlinear modes are taken to be perturbed versions of the linear modes in the absence of internal resonance.
2.1 Literature review of structural mechanics approach to nanotubes

Besides the experimental works on carbon nanotubes, analysis of carbon nanotubes by theoretical modeling has been pursued by many researchers. Classifying the modeling approaches into two categories, one is the atomistic modeling and the other is continuum mechanics modeling. The major techniques under atomistic modeling include tight binding molecular dynamics (TMBD), density functional theory (DFT) and classical molecular dynamics (CMD).

The carbon atoms in a carbon nanotube are connected by strong covalent bonds. No other element in the periodic table bonds to itself in an extended network with the strength of the carbon-carbon bond. These bonds have their characteristic bond length and bond angles in a three-dimensional space. There are similarities between the molecular model of nanotube and the structure of a frame building. The carbon atoms act as joint of the connecting elements. Thus at atomic scale the analysis of mechanical response of carbon nanotubes is demanded. The practical applications of these atomistic
modeling techniques are limited to systems containing fewer numbers of atoms or molecules due to the high computational tasks and expenses.

Ru (5, 6 and 11) proposed a continuum shell model to investigate the buckling properties of carbon nanotubes subject to axial compression. However this model neglects the nanotube chirality and is unable to account for forces acting on the individual atoms. Ru used the elastic shell model to study the effect of van der walls forces on axial buckling of a double walled carbon nanotube. The analysis showed that the van der walls force do not increase the critical axial buckling strain of a double walled carbon nanotube. Thereafter he developed the analysis of multi walled carbon nanotubes based on multiple elastic beam models that considers the interlayer radial displacements coupled through van der walls forces which were ignored earlier by single beam elastic model, and they play a crucial role at ultra high frequencies.

The multi walled carbon nanotube modeled by Ru assumes the concentric nanotubes as individual elastic beams and the deflections of all the nested tubes are coupled through van der walls interaction between any two adjacent nanotubes. There is strong evidence that the interlayer displacements and the associated van der walls forces could have a high effect on the mechanical behavior of multi walled carbon nanotubes. But the interwall van der walls interactions do not affect significantly the elastic moduli of multi walled carbon nanotubes.

Classical continuum mechanics has been resorted for the modeling of carbon nanotubes. For instance, Yakobson et al. (12) noticed the unique features of fullerenes and developed a continuum shell model. Tersoff (7) treated the plane sheet of graphite as an elastic continuum; performed simple calculations of their energies based on their
deformation and concluded that the elastic strain energy of nanotubes can be predicted from the elastic properties of the graphite sheet.

The van der Waals interaction energy as a function of the interlayer spacing between any two adjacent walls of a multi walled carbon nanotube can be estimated using Lennard-Jones potential. Girifalco and Lad (4) computed the lattice summation of the potential energy in the graphite system by assuming Lennard-Jones 6-12 potential between carbon atoms. The strength and effect of van der waals forces on the shape of single and multi walled carbon nanotubes can be investigated using continuum mechanics, and atomic force microscopy. Even though the nanotubes interact only through van der waals forces with the substrate, they experience radial and axial displacements, which significantly modify their concentric geometry.

The intrinsic van der waals attraction among the nanotubes causes them to agglomerate, in concert with their high surface area and aspect ratio. Although the van der waals forces resist the interlayer radial displacements, it is not clear whether the magnitude of the van der waals interaction in carbon nanotubes is strong enough to prevent any significant interlayer radial displacement. Thus the overall buckling behavior of multi walled carbon nanotubes could be affected due to the interlayer radial displacements.

2.2 Mechanical properties of multi walled carbon nanotubes

Carbon nanotubes have been of great interest as mechanically strong materials and for nanometer size electronics. In muti-wall carbon nanotubes, the interlayer interaction
between adjacent layers is sufficiently small compared with the intralayer covalent C-C bonding, so that the sliding motion between the constituent layers is easy if the ends of multi-wall carbon nanotubes are open. Recently, fullerence-encapsulated single wall carbon nanotubes have been produced in which the fullernes can enter from the open end by oxidization.

Carbon nanotubes are predicted to have interesting mechanical properties- in particular, high stiffness and axial strength, as a result of their seamless cylindrical graphitic structure. Their mechanical properties have so far eluded direct measurement, however, because of the very small dimensions of the very small dimensions of the nanotubes. The properties of nanotubes depend upon their atomic arrangement, how the graphite sheets are rolled and the nanostructure.

Depending up on the tube chirality, nanotubes can either be metallic or semi-conducting, as graphite is a semimetal. The mechanical properties of carbon nanotubes have been studied by Qian, Wagner, Liu and Yu (13) using atomic force microscopy, molecular dynamic simulation and empirical force-constant model. The understanding of mechanical response such as yielding behavior is significantly important. The combination of high stiffness and an extraordinary flexibility and resistance to fracture increases the unusual strength of carbon nanotubes.

2.3 Strength of carbon nanotubes

Carbon nanotubes are one of the strongest materials known, both in terms of tensile strength and elastic modulus. This strength results from the covalent sp² bonds formed
between the individual carbon atoms. In 2000, Qing Zhong (16) tested a multi-walled carbon nanotube to have a tensile strength of 63 GPa. In comparison, high carbon steel has a tensile strength of approximately 1.2 GPa. CNTs have very high elastic modulus, in the order of 1 TPa. Since carbon nanotubes have a low density for a solid of 1.3-1.4 g/cm$^3$, its specific strength is the best of known materials.

The mechanical response of carbon nanotubes to severe deformations and strains has attracted much attention since their discovery in 1991. Carbon nanotubes have already demonstrated exceptional mechanical properties: their excellent flexibility during bending has been observed experimentally and studied theoretically. Nanotubes combine high stiffness with resilience and the ability to buckle and collapse in a reversible manner: even largely distorted configurations (axially compressed or twisted) can be due to elastic deformations with virtually no atomic defects involved. For these reasons, it has been suggested that carbon nanotubes could be promising candidates for a new generation of extremely light and super strong fibers.

Theoretically, investigating the ultimate strength of carbon nanotubes requires modeling of inherently mesoscopic phenomena, such as plasticity and fracture, on a microscopic, atomistic level, which presents its own set of challenges. However, the initial stages of strain-induced transformations can be deduced from simulations and these results can be further refined by detailed investigations of the potential energy landscape.

It is now well established from simulations that beyond a certain value of the applied strain, around 5%–6%, single walled carbon nanotubes respond to the mechanical stimuli via the spontaneous formation of topological defects. Given the well-known 1 TPa
Young modulus of graphite and a similar modulus of nanotubes when rescaled to the
density of graphite, the range of elastic response indicates enormous strength, unmatched
by any other known material. It has also been shown that the mechanical response
depends critically on tube geometry: zigzag tubes display a higher strain resistance than
armchair tubes with same diameter. Different orientations of the carbon bonds with
respect to the strain axis in tubes of different symmetry lead to completely different
scenarios: ductile or brittle behaviors can be observed in nanotubes of different indices
under the same external conditions.

Furthermore, the behavior of nanotubes under large tensile strain strongly depends
on their symmetry and diameter and the initial stages of strain-induced transformations
can be explained by dislocation theory. Recently, some of these predictions have been
confirmed by two independent experiments: a number of carbon nanotubes do not exhibit
mechanical failure and breakage up to about 5% strain.

These results, although important in addressing the problem of strength of carbon
nanotubes, do not clearly determine the ultimate limit of their mechanical response, due
to the quality of the samples and experimental limitations. In order to determine the
ultimate limits of the elastic response of strained carbon nanotubes, one needs to address
not only the problem of the stability of topological defects under strain, but equally
importantly, as it turns out, the energetic of the activation process.

Results indicate that although the topological defects become energetically stable at
strains of the order of 5%–6%, the activation barriers for their formation are extremely
high, thus hindering the creation of such defects even at relatively large strains. This
implies that ideal, structurally perfect single-walled carbon nanotubes should be
kinetically stable and resist strains well beyond the 5% observed experimentally and that they can indeed be considered the strongest materials known.

According to Hooke’s law of elasticity, the young’s modulus \( E \) of a material is the ratio of normal stress \( \sigma \) to the normal strain \( \varepsilon \) with in the elastic limit. The value of Young’s modulus depends on the inverse of the wall thickness. The high elasticity of carbon nanotubes are used in the scanning of tips of atomic force microscopy and the elements of nanometer-structured materials. Carbon nanotubes result in a new class of advanced materials because of their mechanical properties. The strength to weight ratio is five hundred times greater than aluminum. From the conventional definition, Young’s modulus is the second derivative of energy \( U \) with respect to the applied strain \( \varepsilon \) divided by the equilibrium volume. A different magnitude is introduced to characterize the stiffness of a carbon nanotube which is independent of any shell thickness given by Hernandez (10) is,

\[
E = \frac{1}{S_0} \left( \frac{\partial^2 U}{\partial \varepsilon^2} \right)_{\varepsilon=0}
\]

where \( S_0 \) is the equilibrium surface of the carbon nanotube.

Kizuka (15) observed that nanotubes can be bent up to a high angle elastically. The shear strength to pull out a nanotube from a bundle can be estimated through shear modulus. The shear modulus of a carbon nanotube is comparable to those of diamond and is also sensitive to tube diameter and tube chirality. If \( L \) is the length of nanotube, \( T \) is the torque acting on the tube, \( \theta \) is the torsion angle and \( J \) is the cross-sectional inertia of the
carbon nanotube, then the shear modulus is given by,  \( G = \frac{TL}{J\theta} \). Shear modulus also increases with the increasing tube diameter.

Another interesting property of carbon nanotubes is poisson's ratio. It can be estimated using Young’s modulus \( E \) and shear modulus \( G \) that is equal to the ratio of the relative tube expansion to the relative axial tube shortening. Poissons ratio measures how much the tube expands or contracts radially when subjected to negative or positive axial strain. Poissons ratio is defined as 

\[
\nu = \frac{E}{2G} - 1
\]

Table 2.1 Elastic properties of multi walled carbon nanotubes by Jian Ping Lu (9)

<table>
<thead>
<tr>
<th>Nanotube</th>
<th>Inner radius (nm)</th>
<th>Outer radius (nm)</th>
<th>Young’s modulus (TPa)</th>
<th>Shear modulus (TPa)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.34</td>
<td>0.68</td>
<td>1.05</td>
<td>0.455</td>
<td>0.270</td>
</tr>
<tr>
<td>2</td>
<td>0.68</td>
<td>1.02</td>
<td>1.08</td>
<td>0.464</td>
<td>0.269</td>
</tr>
<tr>
<td>3</td>
<td>1.02</td>
<td>1.36</td>
<td>1.09</td>
<td>0.472</td>
<td>0.269</td>
</tr>
<tr>
<td>4</td>
<td>1.36</td>
<td>1.70</td>
<td>1.10</td>
<td>0.481</td>
<td>0.269</td>
</tr>
</tbody>
</table>
3.1 Problem Formulation

Beams are widely used in various kinds of engineering structures. Vibration Analysis is essential, especially when the response needs to be accurately determined in order to avoid any resonances caused by internal or external forces.

Fig 2.1 Series of beams connected by linear springs
The $i^{th}$ beam is made of an elastic material of modulus of elasticity $E_i$, mass density $\rho_i$, uniform cross sectional area $A_i$ and moment of inertia about its centroidal axis $I_i$. We assume that $E$, $A$, $I$, $L$, $\rho$ are positive. Let $w_i(x,t)$ correspond to the transverse displacement of the $i^{th}$ beam. The $i^{th}$ beam and the $(i+1)^{th}$ beams are coupled through a flexible layer of stiffness per unit length $c_i$. The outer beams are connected through elastic layers to rigid supports. The first and last beams are connected to rigid foundations through elastic layers of stiffness per unit lengths of $c_0$ and $c_n$ respectively. Assuming uniform properties the set of coupled partial differential equations governing the motion is

$$E_i I_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} - c_0 w_i - c_i (w_2 - w_i) = 0$$

$$E_2 I_2 \frac{\partial^4 w_2}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 w_2}{\partial t^2} + c_1 (w_2 - w_1) - c_2 (w_3 - w_2) = 0$$

$$\vdots$$

$$E_i I_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} + c_{i-1} (w_i - w_{i-1}) - c_i (w_{i+1} - w_i) = 0$$

$$\vdots$$

$$E_{n-1} I_{n-1} \frac{\partial^4 w_{n-1}}{\partial x^4} + \rho_{n-1} A_{n-1} \frac{\partial^2 w_{n-1}}{\partial t^2} + c_{n-2} (w_{n-2} - w_{n-1}) - c_{n-1} (w_n - w_{n-1}) = 0$$

$$E_n I_n \frac{\partial^4 w_n}{\partial x^4} + \rho_n A_n \frac{\partial^2 w_n}{\partial t^2} + c_n w_n + c_{n-1} (w_n - w_{n-1}) = 0$$

(3.1)
3.2 Nondimensionalization

The form of a solution of a partial differential equation can depend critically on the units one chooses for the various quantities involved. These choices can lead to substantial problems frequently when numerical approximation techniques are applied. These difficulties can be avoided by non dimensionalization technique that changes the variables on a paper scale. This technique is useful for analyzing mathematical models.

The non dimensionalization of the governing differential equations can be used to develop independent forms of the independent non dimensional parameters. This method allows the identification of parameters which do not have as much as effect on these dependent parameters. As a result, the number of parameters in such a dimensionless equation is reduced, and this leads to a better understanding of the system being studied. Another advantage of non dimensionalization is that it makes clear which combination(s) of parameters determine the physical behavior of the solution.

Define the non-dimensional parameters as

\[
 w_i^* = \frac{w_i}{L}, i = 1,2,\ldots,n \tag{3.2}
\]

\[
 x^* = \frac{x}{L} \tag{3.3}
\]

\[
 t^* = t \left[ \frac{E_1 I_1}{\rho_1 A_1 L^4} \right] \tag{3.4}
\]
Introducing the non dimensional parameters into equations (3.1) and dropping the *’s leads to

\[
\nu_1 \frac{\partial^4 w_1}{\partial x^4} + \mu_1 \frac{\partial^2 w_1}{\partial t^2} + \phi_1 w_1 - \phi_1 (w_2 - w_1) = 0
\]

\[
\nu_2 \frac{\partial^4 w_2}{\partial x^4} + \mu_2 \frac{\partial^2 w_2}{\partial t^2} + \phi_1 (w_2 - w_1) - \phi_1 (w_3 - w_2) = 0
\]

\[
\nu_3 \frac{\partial^4 w_3}{\partial x^4} + \mu_3 \frac{\partial^2 w_3}{\partial t^2} + \phi_1 (w_3 - w_2) - \phi_1 (w_4 - w_3) = 0
\]

\[
\vdots
\]

\[
\nu_j \frac{\partial^4 w_{j-1}}{\partial x^4} + \phi_{j-1} (w_j - w_{j-1}) - \phi_j (w_{j+1} - w_j) + \mu_j \frac{\partial^2 w_j}{\partial t^2} = 0
\]

\[
\vdots
\]

\[
\nu_{n-1} \frac{\partial^4 w_{n-2}}{\partial x^4} + \phi_{n-2} (w_{n-1} - w_{n-2}) - \phi_{n-1} (w_n - w_{n-1}) + \mu_{n-1} \frac{\partial^2 w_{n-1}}{\partial t^2} = 0
\]

\[
\nu_n \frac{\partial^4 w_n}{\partial x^4} + \lambda_{n-1} (w_n - w_{n-1}) + \phi_n w_n + \mu_n \frac{\partial^2 w_n}{\partial t^2} = 0
\]

(3.5)

Define

\[
\nu_j = \frac{E_j I_j}{E_1 I_1}
\]

(3.6)

\[
\mu_j = \frac{\rho_j A_j}{\rho_1 A_1}
\]

(3.7)

\[
\phi_j = \frac{c_j L^4}{E_1 I_1}
\]

(3.8)

By the method of separation of variables, the dependent variables are assumed to products of functions of independent variables.
A normal mode solution of equation (3.5) is assumed as

$$w_i(x,t) = f_i(x)e^{i\omega t} \quad (3.9)$$

Where $\omega$ is the frequency of vibration and $f = [f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(x)]'$ is the corresponding mode shape vector. Substitution of equation (3.9) into equations (3.5) leads to the set of coupled ordinary differential equations

$$f_1^{(iv)} - \omega^2 f_1 + \phi_0 f_1 - \phi_1 (f_2 - f_1) = 0$$

$$f_2^{(iv)} - \omega^2 \frac{\mu_2}{\nu_2} f_2 + \frac{\phi_1}{\nu_2} (f_2 - f_1) - \frac{\phi_2}{\nu_2} (f_3 - f_2) = 0$$

$$f_3^{(iv)} - \omega^2 \frac{\mu_3}{\nu_3} f_3 + \frac{\phi_2}{\nu_3} (f_3 - f_2) - \frac{\phi_3}{\nu_3} (f_4 - f_3) = 0$$

$$\vdots$$

$$f_j^{(iv)} - \omega^2 \frac{\mu_j}{\nu_j} f_j + \frac{\phi_{j-1}}{\nu_j} (f_j - f_{j-1}) - \frac{\phi_j}{\nu_j} (f_{j+1} - f_j) = 0$$

$$\vdots$$

$$f_{n-1}^{(iv)} - \omega^2 \frac{\mu_{n-1}}{\nu_{n-1}} f_{n-1} + \frac{\phi_{n-2}}{\nu_{n-1}} (f_{n-1} - f_{n-2}) - \frac{\phi_{n-1}}{\nu_{n-1}} (f_n - f_{n-1}) = 0$$

$$f_n^{(iv)} - \omega^2 \frac{\mu_n}{\nu_n} f_n + \frac{\phi_{n-1}}{\nu_n} (f_n - f_{n-1}) + \lambda_n f_n = 0 \quad (3.10)$$

Let $z(x)$ satisfy the differential equation

$$z^{(iv)}(x) = \delta^2 z(x) \quad (3.11)$$

and also the boundary conditions for each beam. The $\delta$ values correspond to the eigenvalues of the problem and are ordered according to $0 \leq \delta_1 < \delta_2 < \ldots < \delta_{k-1} < \delta_k < \delta_{k+1} < \ldots$. For each $\delta$ there is a corresponding $z_k(x)$.
A solution to equations (3.10) is assumed as

\[ f = a * z(x) \]  \hspace{1cm} (3.12)

Substitution of equation (3.12) in to equations (3.10) leads to the matrix eigen value problem

\[ Ka = \omega^2 Ma \]  \hspace{1cm} (3.13)

where \( K \) is a tri-diagonal matrix whose elements are

\[
k_{i,i} = \delta^2 + \frac{\phi_{i-1}}{v_i} + \phi_i \quad i = 1,2,\ldots,n
\]

\[
k_{i,i-1} = -\phi_{i-1} \quad i = 2,3,\ldots,n
\]

\[
k_{i,i+1} = -\phi_i \quad i = 1,2,\ldots,n-1
\]  \hspace{1cm} (3.14)

and \( M \) is a diagonal matrix with

\[
m_{i,i} = \mu_i
\]  \hspace{1cm} (3.15)

The forms of \( z_k(x) \) and the first five values of \( \delta \) for specific end conditions are given in Table 3.1

Table 3.1 Natural frequencies and mode shapes for beams (3).

<table>
<thead>
<tr>
<th>End Condition</th>
<th>Characteristic Equation</th>
<th>Five lowest Frequencies</th>
<th>Mode shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = 0; x = 1 )</td>
<td>natural ( z(x) )</td>
<td>( \omega_1 = 22.37 )</td>
<td></td>
</tr>
<tr>
<td>( \omega_2 = 61.66 )</td>
<td>( \omega_3 = 120.9 )</td>
<td>( \omega_4 = 199.9 )</td>
<td></td>
</tr>
<tr>
<td>( \omega_5 = 298.6 )</td>
<td>( \omega_6 = 372.2 )</td>
<td>( \omega_7 = 566.6 )</td>
<td></td>
</tr>
</tbody>
</table>

Fixed-fixed \( \cos \frac{\lambda}{2} \cosh \frac{\lambda}{2} = 1 \)

\[
\omega_1 = \frac{\cosh \lambda}{\lambda \sinh \lambda} \left( \cosh \frac{\lambda}{2} - \cot \frac{\lambda}{2} \right)
\]

\[
\omega_2 = \frac{\cosh \lambda}{\lambda \sinh \lambda} \left( \cosh \frac{\lambda}{2} - \cot \frac{\lambda}{2} \right)
\]

\[
\omega_3 = \frac{\cosh \lambda}{\lambda \sinh \lambda} \left( \cosh \frac{\lambda}{2} - \cot \frac{\lambda}{2} \right)
\]

\[
\omega_4 = \frac{\cosh \lambda}{\lambda \sinh \lambda} \left( \cosh \frac{\lambda}{2} - \cot \frac{\lambda}{2} \right)
\]

\[
\omega_5 = \frac{\cosh \lambda}{\lambda \sinh \lambda} \left( \cosh \frac{\lambda}{2} - \cot \frac{\lambda}{2} \right)
\]
Table 3.1 Natural frequencies and mode shapes for beams (3) (continued).

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Pinned-pinned</td>
<td>$\sin \frac{\lambda}{4} = 0$</td>
<td>$\omega_1 = 9.870$</td>
<td>$C_4 \sin \frac{\lambda}{4}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = 39.48$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_3 = 88.83$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_4 = 157.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_5 = 246.7$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed-free</td>
<td>$\cos \frac{\lambda}{4} \cosh \frac{\lambda}{4} = -1$</td>
<td>$\omega_1 = 3.51$</td>
<td>$C_4 \left[ \cosh \frac{\lambda}{4} x - \cos \frac{\lambda}{4} x - \alpha \sin \frac{\lambda}{4} x - \sin \frac{\lambda}{4} x \right]$</td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = 22.03$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_3 = 61.07$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_4 = 120.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_5 = 199.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Free-free</td>
<td>$\cosh \frac{\lambda}{4} \cos \frac{\lambda}{4} = 1$</td>
<td>$\omega_1 = 0$</td>
<td>$C_4 \left[ \cosh \frac{\lambda}{4} x + \cos \frac{\lambda}{4} x + \alpha \sin \frac{\lambda}{4} x + \sin \frac{\lambda}{4} x \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = 22.37$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_3 = 61.66$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_4 = 120.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_5 = 199.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pinned-free</td>
<td>$\tan \frac{\lambda}{4} = \tanh \frac{\lambda}{4}$</td>
<td>$\omega_1 = 0$</td>
<td>$C_4 \left[ \sin \frac{\lambda}{4} x + \sin \frac{\lambda}{4} x \cosh \frac{\lambda}{4} - \sin \frac{\lambda}{4} x \cosh \frac{\lambda}{4} \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = 15.42$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_3 = 49.96$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_4 = 104.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_5 = 178.3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed-pinned</td>
<td>$\tan \frac{\lambda}{4} = \tanh \frac{\lambda}{4}$</td>
<td>$\omega_1 = 15.42$</td>
<td>$C_4 \left[ \cosh \frac{\lambda}{4} x - \cos \frac{\lambda}{4} x - \alpha \sin \frac{\lambda}{4} x - \sin \frac{\lambda}{4} x \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = 49.96$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_3 = 104.2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_4 = 178.3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\omega_5 = 272.0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There are a set of $n$ natural frequencies for each $\delta_k$ which are the square roots of the eigen values of $M^{-1}K$. These are denoted by

$$0 \leq \omega_{k,1} \leq \omega_{k,2} \leq \ldots \leq \omega_{k,n} \quad (3.16)$$
Each natural frequency has a corresponding mode shape of the form
\[ f_{k,j} = a_{k,j} z_k(x) \]
where \( a_{k,1}, a_{k,2}, \ldots, a_{k,n} \) are the eigenvectors of equation (3.13) corresponding to \( \delta_k \). Since \( M \) and \( K \) are symmetric these eigenvectors satisfy an orthogonality condition

\[ a_{k,j}^T M a_{k,j} = 0 \quad i \neq j \]  
(3.17)

The standard inner product on \( \mathbb{R}^n \) is defined as \( (p, q)_n = q^T p \) where \( p \) and \( q \) are arbitrary elements of \( \mathbb{R}^n \). Since \( M \) is symmetric and positive definite with respect to the standard inner product an energy inner product defined by

\[ (p, q)_{M_n} = (M p, q)_n \]  
(3.18)

is also a valid inner product on \( \mathbb{R}^n \). The matrix \( K \) is symmetric and positive definite unless \( \delta = 0 \) and \( \lambda_0 = \lambda_n = 0 \). In this case \( K \) is non-negative definite. When \( K \) is positive definite an energy inner product of the form

\[ (p, q)_{K_n} = (K p, q)_n \]  
(3.19)

may be defined on \( \mathbb{R}^n \). The matrix operator \( M^{-1}K \) is positive-definite and self adjoint with respect to the energy inner products of Equations (3.18) and (3.19). Thus their eigenvalues are real and positive leading to equation (3.20) and their eigenvectors are mutually orthogonal with respect to the energy inner product leading to equation (3.21) which is also written as

\[ (a_{k,i}, a_{k,j})_{M_n} = 0 \quad i \neq j \]  
(3.20)
Let $W$ be the subspace of $C^4[0, 1]$ of all functions that satisfy the boundary conditions for a specific system. Let $V$ be the vector space $W \times \mathbb{R}_n$. The members of $W$ are $n$-dimensional vectors whose elements are functions that are four times differentiable on the domain from 0 to 1 inclusive and satisfy the boundary conditions for a specific beam. The solutions of equations (3.10) are members of $W$. The system of equations (3.10) can be formulated in operator form as

$$Lf = \omega^2 Mf$$  
(3.21)

where

$$Lf = A^{-1}f^{(iv)} + Bf,$$  
(3.22)

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}, \quad f^{(iv)} = \begin{bmatrix} f_1^{(iv)} \\ f_2^{(iv)} \\ \vdots \\ f_{n-1}^{(iv)} \\ f_n^{(iv)} \end{bmatrix}$$  
(3.23)

$A$ is a $n \times n$ diagonal matrix with $a_{ii} = \mu_i \quad i = 1, 2, \ldots, n$ and $B$ is a symmetric tri-diagonal matrix with

$$b_{i,i} = \phi_{i-1} + \phi_i \quad i = 1, 2, \ldots, n$$

$$b_{i,i-1} = -\phi_{i-1} \quad i = 2, 3, \ldots, n$$

$$b_{i,i+1} = -\phi_i \quad i = 1, 2, \ldots, n-1$$  
(3.24)

The natural frequencies are thus the square roots of the eigenvalues of $M^{-1}L$ and the mode shape vectors are the corresponding eigenvectors.
The general solution for the transverse displacement is obtained using equations

(3.9) and (3.10) as

\[ W(x,t) = \sum_{k=1}^{n} \sum_{l=1}^{n} \left[ a_{k,l} \right] A_{k,l} \phi_k(x)e^{j\omega t} + \text{C.C.} \]  

(3.25)

where the \( A_{k,l} \) are the constants of integration and \( \text{C.C.} \) stands for complex conjugate.

3.3 System with quadratic nonlinearity

Consider a system in which the elastic layers between the beams is nonlinear with a force per unit length between the layers ‘j’ and ‘j-1’ given by

\[ F_j = c_{j,1}(w_j - w_{j-1}) + c_{j,2}(w_j - w_{j-1})^2 \]  

(3.26)

With the application of conservation laws to appropriate beam leads to

\[ E_1 I_1 \frac{\partial^4 w_1}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 w_1}{\partial t^2} - c_0 w_1 - c_{11}(w_2 - w_1) - c_{12}(w_2 - w_1)^2 = 0 \]

\[ E_2 I_2 \frac{\partial^4 w_2}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 w_2}{\partial t^2} + c_{11}(w_2 - w_1) + c_{12}(w_2 - w_1)^2 - c_{21}(w_3 - w_2) - c_{22}(w_3 - w_2)^2 = 0 \]

\[ E_3 I_3 \frac{\partial^4 w_3}{\partial x^4} + \rho_3 A_3 \frac{\partial^2 w_3}{\partial t^2} + c_{21}(w_3 - w_2) + c_{22}(w_3 - w_2)^2 - c_{31}(w_4 - w_3) - c_{32}(w_4 - w_3)^2 = 0 \]

\[ \vdots \]

\[ E_i I_i \frac{\partial^4 w_i}{\partial x^4} + \rho_i A_i \frac{\partial^2 w_i}{\partial t^2} + c_{i-11}(w_i - w_{i-1}) + c_{i-12}(w_i - w_{i-1})^2 - c_{i1}(w_{i+1} - w_i) + c_{i3}(w_{i+1} - w_i)^2 = 0 \]

\[ \vdots \]

30
\[ E_{n-1}I_{n-1} \frac{\partial^4 w_{n-1}}{\partial x^4} + \rho_{n-1}A_{n-1} \frac{\partial^2 w_{n-1}}{\partial t^2} + c_{n-21}(w_{n-1} - w_{n-2}) + c_{n-22}(w_{n-1} - w_{n-2})^2 - c_{n-11}(w_{n-1} - w_n) - c_{n-13}(w_n - w_{n-1})^2 = 0 \]

\[ E_n I_n \frac{\partial^4 w_n}{\partial x^4} + \rho_n A_n \frac{\partial^2 w_n}{\partial t^2} + c_n w_n + c_{n-11}(w_n - w_{n-1}) - c_{n-12}(w_n - w_{n-1})^2 = 0 \]  

(3.27)

where all variables and parameters are defined earlier.

Non-dimensionalization of equation (3.27) leads to:

\[ \frac{\partial^4 w_j}{\partial x^4} + \frac{\mu_j}{\nu_j} \frac{\partial^4 w_j}{\partial t^2} + \frac{\phi_{j-1}}{\nu_j} (w_j - w_{j-1}) + \frac{\beta_{j-1}}{\nu_j} (w_j - w_{j-1})^2 - \frac{\phi_j}{\nu_j} (w_{j+1} - w_j) - \frac{\beta_j}{\nu_j} (w_{j+1} - w_j)^2 = 0 \]

\[ j = 1, 2, \ldots, n \]  

(3.28)
CHAPTER IV

SYSTEM MODEL WITH QUADRATIC NONLINEARITY

4.1 Method of multiple scales

Over the past three decades, we have acquired new tools and techniques to synthesize nanoscale objects and learn their many incredible properties. Advances in the synthesis of nanoscale materials have stimulated ever broader research activities in science and engineering devoted entirely to these materials and their applications. This is due in large part to the combination of their expected structural perfection, small size, low density, high stiffness, high strength and excellent electronic properties. As a result, nanoscale materials may find use in a wide range of applications in material reinforcement, field emission panel display, chemical sensing, drug delivery, nanoelectronics and tailor-designed materials.

While microscale and nanoscale systems and processes are becoming more viable for engineering applications, our knowledge of their behavior and our ability to model their performance remains limited. Continuum based computational capabilities are obviously not applicable over the full range of operational conditions of these devices. Noncontinuum behavior is observed in large deformation behavior of nanotubes, ion deposition processes, gas dynamic transport, and material mechanics as characteristic
scales drop toward the micron scale. At the scales of nano devices, interactions between thermal effects and mechanical response can become increasingly important.

Furthermore, nanoscale components will be used in conjunction with components that are larger and respond at different timescales. In such hybrid systems, the interaction of different time and length scales may play a crucial role in the performance of the complete system. Single scale methods such as ab initio methods or molecular dynamics (MD) would have difficulty in analyzing such hybrid structures owing to the large range of timescales and length scales. For the design and study of nanoscale materials and devices in microscale systems, models must span length scales from nanometers to hundreds of microns.

Multiple scale modeling methods have recently emerged as the tool of choice to link the mechanical behavior of materials from the smallest scale of atoms to the largest scale of structures. Multiple scale methods offer the best hope for bridging the traditional gap that exists between experimental approach, the theoretical approach and computational modeling for studying and understanding the behavior of materials. Multiple scale methods generally imply the utilization of information at one length scale to subsequently model the response of the material at larger length scales.

An asymptotic expansion for \( w(x,t) \) is assumed as

\[
  w_j(x,t) = w_{j0}(x,t) + \varepsilon w_{j1}(x,t) + \varepsilon^2 w_{j2}(x,t) + \varepsilon^3 w_{j3}(x,t) + ....
\]

The expansion is not necessarily convergent, but must be uniformly valid. Each term in equation (4.1) is a function of multiple scales. The method of multiple scales is
employed by defining $T_n = \varepsilon^n t$ where $T_n$ represents different time scales, $T_0$ being the fast time scale and $T_1$ the slow time scale.

$$w(x, t) = w(x, T_0, T_1, T_2, T_3, \ldots T_n) \quad (4.2)$$

The derivatives with respect to $t$ are,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial w}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial w}{\partial T_2} \frac{\partial T_2}{\partial t} + \frac{\partial w}{\partial T_3} \frac{\partial T_3}{\partial t} + \ldots$$

$$= \frac{\partial w}{\partial T_0} + \varepsilon \frac{\partial w}{\partial T_1} + \varepsilon^2 \frac{\partial w}{\partial T_2} + \varepsilon^3 \frac{\partial w}{\partial T_3} + \ldots \quad (4.3)$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial T_0^2} + \varepsilon \left( 2 \frac{\partial^2 w}{\partial T_0 \partial T_1} \right) + \varepsilon^2 \left( \frac{2 \partial^2 w}{\partial T_0 \partial T_2} + \frac{\partial^2 w}{\partial T_1^2} \right) + \ldots \quad (4.4)$$

Substituting equation (4.4) in to (3.28)

$$\frac{\partial^4 w_j}{\partial x^4} + \frac{\phi_{j-1}}{v_j} (w_j - w_{j-1}) + \frac{\beta_{j-1}}{v_j} (w_j - w_{j-1})^2 - \frac{\phi_j}{v_j} (w_{j+1} - w_j) - \frac{\beta_j}{v_j} (w_{j+1} - w_j)^2 + \frac{\mu_j}{v_j} \frac{\partial^2 w_j}{\partial T_0^2}$$

$$+ \frac{\mu_j}{v_j} \frac{\partial^2 w_j}{\partial T_0} + \frac{\mu_j}{v_j} \varepsilon \left( \frac{2 \partial^2 w}{\partial T_0 \partial T_1} \right) + \frac{\mu_i}{v_j} \varepsilon^2 \left( \frac{2 \partial^2 w}{\partial T_0 \partial T_2} + \frac{\partial^2 w}{\partial T_1^2} \right) + \ldots = 0 \quad (4.5)$$

Equation (4.5) must hold for all values of $\varepsilon$ since it is an independent dimensionless parameter. This is true only if coefficients of powers of $\varepsilon$ are independently set to zero.

This leads to

**Order $\varepsilon^0$**

$$\frac{\partial^4 w_{j0}}{\partial x^4} + \frac{\phi_{j-1}}{v_j} (w_{j,0} - w_{j-1,0}) - \frac{\phi_j}{v_j} (w_{j+1,0} - w_{j,0}) + \frac{\mu_j}{v_j} \frac{\partial^2 w_{j0}}{\partial T_0^2} = 0 \quad (4.6)$$
Order $\varepsilon^1$

\[
\frac{\partial^4 w_{j1}}{\partial x^4} + \frac{\phi_{j-1}}{v_j} (w_{j,1} - w_{j-1,1}) \frac{\phi_j}{v_j} (w_{j+1,1} - w_{j,1}) + \frac{\mu_j}{v_j} \frac{\partial^2 w_{j1}}{\partial T_0^2} = -\frac{\beta_{j-1}}{v_j} (w_{j,0} - w_{j-1,0})^2 \\
+ \frac{\beta_j}{v_j} (w_{j+1,0} - w_{j,0})^2 - 2 \frac{\mu_j}{v_j} \left( \frac{\partial^2 w_{j0}}{\partial T_0 \partial T_1} \right)
\] (4.7)

4.2 Zeroth order solution

The solution of equation (4.6) is the solution obtained in equation (3.25) and is

\[
w_{j,0} = \sum_{k=1}^{\infty} \sum_{i=1}^{n} A_{k,i}(T_1)[a_{k,i}]_j \phi_k(x)e^{i\omega_j T_0} + C.C
\] (4.8)

Where C.C refers to complex conjugate.

4.3 First order solution

Substituting equation (4.8) in to equation (4.7) gives

\[
\frac{\partial^4 w_{j1}}{\partial x^4} + \frac{\phi_{j-1}}{v_j} (w_{j,1} - w_{j-1,1}) - \frac{\phi_j}{v_j} (w_{j+1,1} - w_{j,1}) + \frac{\mu_j}{v_j} \frac{\partial^2 w_{j1}}{\partial T_0^2} =
\]

\[
-\frac{\beta_{j-1}}{v_j} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} A_{k,i}(T_1)[a_{k,i}]_j - [a_{k,i}]_{j-1} \phi_k(x)e^{i\omega_j T_0} + C.C \right)^2
\]

\[
+ \frac{\beta_j}{v_j} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} A_{k,i}(T_1)[a_{k,i}]_{j+1} - [a_{k,i}]_j \phi_k(x)e^{i\omega_j T_0} + C.C \right)^2
\]
\[-2 \frac{\mu_j}{v_j} \left( \frac{\partial^2}{\partial T_0 \partial T_1} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} A_{k,i}(T_1) [a_{k,i}] \phi_k(x) e^{i\omega_k T_0} + \text{C.C.} \right) \right) \] (4.9)

After considerable algebra the right hand side of equation (4.9) can be simplified to

\[G_j(x, T_0, T_1) = -2i \sum_{k=1}^{\infty} \sum_{i=1}^{n} \omega_{k,i} A_{k,i}(T_1) [a_{i,j}] \phi_k(x) \alpha_{k,i} e^{i\omega_k T_0} + \text{C.C.} \]

\[+ \sum_{k=p=1}^{\infty} \sum_{i,q=1}^{n} \gamma_{kpiq} \xi_{kpm} \left[ A_{k,i} A_{pq} e^{i(\omega_k + \omega_q) T_0} + \text{C.C.} \right] \] (4.10)

where,

\[\xi_{kpm} = \int_0^1 \phi_k(x) \phi_p(x) \phi_m(x) dx \] (4.11)

\[\gamma_{kpiq} = a_{ki,j+1} a_{pq,j+1} - a_{kij} a_{pg} \] (4.12)

\[v_{kij} = \left[ a_{k,i} \right]_j - \left[ a_{k,i} \right]_{j-1} \] (4.13)

4.4 Modal analysis

The expansion theorem is used to expand \(G\) into a series in terms of normalized eigen function of the zeroth-order solution.

\[G = \sum_{k=1}^{\infty} \sum_{i=1}^{n} h_{k,i}(T_0, T_1) [a_{k,i}] \phi_k(x) \] (4.14)

where

\[h_{i,j}(T_0, T_1) = \int_0^1 [a_{i,j}]^T [G(x,t)] \phi_j(x) dx \]
\[ = \sum_{u=1}^{n} \left[ a_{i,j} \int_{0}^{1} [G_{u}(x,t)] \phi_{i}(x) dx \right] \tag{4.15} \]

A general form of \( h_{i,j}(T_{0},T_{1}) \) is

\[ h_{i,j}(T_{0},T_{1}) = -2i \sum_{k=1}^{\infty} \sum_{l=1}^{n} \omega_{kl} \dot{A}_{kl}(T_{1}) \left[ a_{i,j} \right] \phi_{k}(x) \alpha_{kl} e^{i\omega_{kl}T_{0}} + C.C \]

\[ \sum_{k,p=1}^{\infty} \sum_{i,q=1}^{n} \gamma_{kpiq} \xi_{kpm} \left[ A_{kl} A_{pq} e^{i(\omega_{kl} + \omega_{pq})T_{0}} + C.C \right] \tag{4.16} \]

The expansion theorem is used to express \( w \) as

\[ w = \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right] \phi_{k}(x) \tag{4.17} \]

Substituting equation (4.17) into (4.9)

\[ \frac{\partial^{4}}{\partial x^{4}} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right] \phi_{k}(x) \right) \]

\[ + \frac{\phi_{j-1}}{v_{j}} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right]_{j-1,j} \phi_{k}(x) - \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right]_{j-1,j} \phi_{k}(x) \right) \]

\[ - \frac{\phi_{j}}{v_{j}} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right]_{j+1,j} \phi_{k}(x) - \sum_{k=1}^{\infty} \sum_{i=1}^{n} b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right]_{j+1,j} \phi_{k}(x) \right) \]

\[ + \frac{\mu}{v_{j}} \frac{\partial^{2}}{\partial T_{0}^{2}} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \left( b_{k,i}(T_{0},T_{1}) \left[ a_{k,i} \right]_{j} \phi_{k}(x) \right) \]

\[ = -2i \sum_{k=1}^{\infty} \sum_{l=1}^{n} \omega_{kl} \dot{A}_{kl}(T_{1}) \left[ a_{i,j} \right] \phi_{k}(x) \alpha_{kl} e^{i\omega_{kl}T_{0}} + C.C \]

\[ + \sum_{k,p=1}^{\infty} \sum_{i,q=1}^{n} \gamma_{kpiq} \xi_{kpm} \left[ A_{kl} A_{pq} e^{i(\omega_{kl} + \omega_{pq})T_{0}} + C.C \right] \tag{4.18} \]
Taking the inner product of equation (4.18) with respect to an arbitrary mode shape

\[ \mathbf{f}_{m,n} = \mathbf{a}_{m,n} \phi_m(x) \] leads to

\[
\frac{\partial^4}{\partial x^4} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_i \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx + \\
\phi_{j-1} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_i \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx - \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_{j-1} \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx \right) \\
- \frac{\phi_j}{V_j} \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_{j-1} \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx - \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_{j-1} \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx \right) \\
+ \frac{\mu_j}{V_j} \frac{\partial^2}{\partial T_0^2} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} b_{k,i}(T_0, T_1) \cdot [a_{k,i}]_{j-1} \cdot \phi_k(x) \cdot [a_{m,n}]_m \cdot \phi_m(x) \cdot dx \\
= \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{0}^{1} h_{k,i}(T_0, T_1) \cdot [a_{k,i}]_i \cdot \phi_k(x) \cdot dx \quad (4.19)
\]

Using orthonormality condition equation (4.19) becomes

\[
\frac{\partial^2 b_{m,n}}{\partial T_0^2} + \omega_{mn}^2 b_{m,n} = h_{m,n}(T_0, T_1) \quad i = 1,2,...,\infty \\
j = 1,2,...,n 
\quad (4.20)
\]

Equation (4.20) represents a set of uncoupled linear ordinary differential equations to solve for \( b_{m,n}(T_0, T_1) \). The solution for \( b_{m,n}(T_0, T_1) \) leads to a uniform expansion only if this linear combination does not include any terms proportional to \( e^{i\omega_{mn}T_0} \). Terms in which the frequencies are near to \( \omega_{mn} \) are called secular terms. The expansion is rendered uniform only if all secular terms are removed.
5.1 Internal resonances

One of the most important features of nonlinear systems with several degrees of freedom is the presence of internal resonances at certain relations between natural frequencies of different modes. The response of a system depends on the order of its nonlinearity and its internal resonances. The internal resonances that might occur in a system depend on its geometry, composition, and the boundary conditions. An \( n \)-degree of freedom system has \( n \) linear natural frequencies and \( n \) corresponding modes. The natural frequencies are denoted by \( \omega_1, \omega_2, ..., \omega_n \) and assuming that they are real and different from zero. An important case occurs whenever two or more are commensurable or nearly commensurable. Examples of near-commensurability are

\[
\omega_{ij} \approx 2 \omega_{pq}, \quad \omega_{ij} \approx \omega_{pq} \pm \omega_{rs},
\]

If \( \omega_{pq} \pm \omega_{rs} \approx \omega_{ij} \), then \( \omega_{pq} \pm \omega_{rs} = \omega_{ij} + \varepsilon \sigma \), where \( \sigma \) is called the detuning parameter. Depending on the order of nonlinearity in the system, these commensurable relationships of frequencies can cause the corresponding modes to be strongly coupled, and an internal resonance is said to exist. If the system has quadratic nonlinearities, then
to first order an internal resonance can exist if $\omega_m \approx 2\omega_k$ or $\omega_q \approx \omega_p \pm \omega_m$. The type of combination resonance which might exist in a system depends on the order of nonlinearity in the system. For a system having quadratic nonlinearities in which $\omega_2 \approx 2\omega_1$, Nayfeh, Mook, and Marshal (2) showed that a saturation phenomenon exists when the harmonic external excitation of frequency is approximately equal to $\omega_2$. When the amplitude of excitation $k$ is small, only the second mode with frequency $\omega_2$ is excited.

As $k$ reaches a critical value $k_c$, which depends on the damping coefficients of the two modes and the detunings, this mode saturates and the first mode begins to grow. As $k$ increases further, all the additional energy goes to the first mode (even though the harmonic external excitation of frequency is approximately equal to $\omega_2$) due to the internal resonance. When internal resonances exists in a free system, energy imparted initially to one of the modes involved in internal resonance will be continuously exchanged among all the modes in that internal resonance.

The presence of more than one natural frequency and mode produces physical phenomena in nonlinear systems such as internal resonances, combinational resonances, saturation and the nonexistence of periodic responses to a periodic excitation in the presence of positive damping. These physical phenomena are the characteristics of nonlinear multi-degree-of-freedom systems. Equation (4.9) represents the mode shapes of the first-order solution. The expansion $w(x,t) = w_0(x,t) + \varepsilon w_1(x,t) + \ldots$ is uniform if $w_1(x,t)$ is bounded compared to $w_0(x,t)$. The right hand side of equation (4.9) is a
combination of harmonic terms at different frequencies. Equation (4.18) shows that each equation has a secular term of

$$-2iA_{i,j}(T)e^{i\omega_i T}$$

(5.2)

Additional secular terms in each equation arise corresponding to frequencies which satisfy

$$\omega_{p,q} \pm \omega_{r,s} \approx \omega_{i,j}$$

(5.3)

When frequencies satisfy conditions of the form of equation (5.3) an internal resonance occurs. Elimination of secular terms from each equation requires elimination of internal resonances.

5.2 Lennard-Jones potential

The van der Waals interaction energy potential estimated by the Lennard-Jones pair potential, as a function of the inter atomic interaction between two adjacent nanotubes is given by (4),

$$\phi(d) = \frac{A}{d^6} \left( \frac{1}{2} \frac{d_0^6}{d^{12}} - \frac{1}{d^6} \right)$$

(5.4)

where the interatomic distance $d$ is normalized by the carbon-carbon bond length $a = 0.142 nm$, the van der Waals distance $d_0$ and the energy constant $A$ are taken, respectively, to be 2.7 and 24.3 x 10^{-79} Jm^6.

Let $y = \frac{d}{a}$, the above equation becomes,
\[ \phi(y) = \frac{A}{a^6} \left( \frac{1}{2} \frac{y_0^6}{y^{12}} - \frac{1}{y^6} \right) \quad (5.5) \]

Girifalco and Lad (4), obtained the value of \( y_0 \), the equilibrium distance, from the condition that at equilibrium,

\[ \left( \frac{\partial \phi_{x,\infty}}{\partial y} \right)_{\text{equilibrium}} = 0 \quad (5.6) \]

The equilibrium distance for each potential function is different and is obtained by plots of \( \phi(y) \). For a hypothetical case, the equilibrium distance between two carbon atoms is 2.70. The equilibrium distance decreases as expected because of the increased number of atoms interacting with the atom above the graphite sheet. Thus for a potential energy function between two semi infinite graphite sheets, the distance is measured to be 2.36, in units of the carbon-carbon distance in a graphite monolayer, in the region of the minimum from the plot.

Let \( S^{m}_{x,\infty}(x) \) be the summation of \( 1/y_i^m \) terms where \( x \) is the distance between an atom above the center of the hexagon and the graphite sheet.

Girifalco and Lad (4), obtained the potential energy function for interaction between two semi infinite graphite sheets,

\[ \phi_{x,\infty}(x) = \frac{A}{2\sigma a^6} \left( \frac{1}{2} a_0^6 S^{12}_{x,\infty}(x) - S^6_{x,\infty}(x) \right) \quad (5.7) \]

where \( 1/2\sigma \) denotes the number of atoms of each type per centimeter square.
Girifalco and Lad (4), presented numerical computation of $S_{\alpha,\infty}^{12}(x)$ and $S_{\alpha,\infty}^{6}(x)$ for discrete values of $x$, and used their numerical data to determine a continuous model. An alternate representation of a continuous model is obtained by assuming

$$S_{\alpha,\infty}^{12}(x) = a_1 x^{b_1} \quad (5.8)$$

$$S_{\alpha,\infty}^{6}(x) = a_2 x^{b_2} \quad (5.9)$$

The coefficients $a_1, b_1, a_2$ and $b_2$ are determined using a linear least squares method, resulting in,

$$S_{\alpha,\infty}^{12}(x) = \frac{1.04}{x^{0.077}}, \quad (5.10)$$

$$S_{\alpha,\infty}^{6}(x) = \frac{2.182}{x^{3.59}} \quad (5.11)$$

Substitution of equations (5.10) and (5.11) into equation (5.7) leads to

$$\phi_{\alpha,\infty}(x) = \frac{A}{2\sigma d^6} \left( \frac{1}{2} d_0^6 \left( \frac{1.04}{x^{10.077}} \right) - \frac{2.182}{x^{3.59}} \right) \quad (5.12)$$

At equilibrium, $x = 2.36$, gives

$$\phi_{\alpha,\infty}(2.36) = \frac{24.3 \times 10^{-60}}{2(2.62 \times 10^{-16})(0.142 \times 10^{-9})^6} \left( \frac{1}{2} \frac{1.04}{2.36^{10.077}} - \frac{2.182}{2.36^{3.59}} \right)$$

$$\phi_{\alpha,\infty}(2.36) = 366.67 \text{ ergs/cm}^2 \quad (5.13)$$

When the system is in equilibrium the potential energy is a minimum. When the separation distance between atoms is changed then the potential energy increases. Let $x = x_0 + \delta$ where $x_0 = 2.36$ is the equilibrium distance and $\delta$ is a small displacement.
Equation (5.13) leads to

\[
\phi_{x_e}(x_0 + \delta) = \frac{A}{2\sigma d^6} \left\{ \frac{1}{2} d^6_0 \left( 1.04(x_0 + \delta)^{-10.077} - 2.182(x_0 + \delta)^{-3.59} \right) \right\}
\]

\[
\phi_{x_e}(x_0 + \delta) = \frac{A}{2\sigma d^6} \left\{ \frac{1}{2} d^6_0 \left( 1.04 x_0^{-10.077} \left( 1 + \frac{\delta}{x_0} \right)^{-10.077} - 2.182 x_0^{-3.59} \left( 1 + \frac{\delta}{x_0} \right)^{-3.59} \right) \right\} \tag{5.14}
\]

Assuming \( \frac{\delta}{x_0} < 1 \), binomial expansions are used in equation (5.14)

\[
\left(1 + \frac{\delta}{x_0}\right)^{-10.077} = 1 - 10.077 \left( \frac{\delta}{x_0} \right) + \frac{10.077 \cdot 0.1077}{2} \left( \frac{\delta}{x_0} \right)^2 - \frac{10.077 \cdot 0.1077 \cdot 0.2077}{6} \left( \frac{\delta}{x_0} \right)^3 + ...
\]

\[
= 1 - 10.077 \left( \frac{\delta}{x_0} \right) + 55.81 \left( \frac{\delta}{x_0} \right)^2 - 224.68 \left( \frac{\delta}{x_0} \right)^3 + ... \tag{5.15}
\]

\[
\left(1 + \frac{\delta}{x_0}\right)^{-3.59} = 1 - 3.59 \left( \frac{\delta}{x_0} \right) + \frac{3.59 \cdot 4.59}{2} \left( \frac{\delta}{x_0} \right)^2 - \frac{3.59 \cdot 4.59 \cdot 5.59}{6} \left( \frac{\delta}{x_0} \right)^3 + ...
\]

\[
= 1 - 3.59 \left( \frac{\delta}{x_0} \right) + 8.24 \left( \frac{\delta}{x_0} \right)^2 - 15.352 \left( \frac{\delta}{x_0} \right)^3 + ... \tag{5.16}
\]

Substituting (5.14) and (5.15) into (5.13) leads to

\[
\phi_{x_e}(x_0 + \delta) = \frac{A}{2\sigma d^6} \left\{ \frac{1}{2} d^6_0 \left( 1.04 x_0^{-10.077} \left( 1 - 10.077 \left( \frac{\delta}{x_0} \right) + 55.81 \left( \frac{\delta}{x_0} \right)^2 - 224.68 \left( \frac{\delta}{x_0} \right)^3 + ...ight) \right) \right\}
\]

\[
- 2.182 x_0^{-3.59} \left( 1 - 3.59 \left( \frac{\delta}{x_0} \right) + 8.24 \left( \frac{\delta}{x_0} \right)^2 - 15.352 \left( \frac{\delta}{x_0} \right)^3 + ... \right) \right\} \tag{5.17}
\]

The force per unit length between two adjacent nanotubes that have been separated by \( \delta \) from the equilibrium distance is
\[ F = \frac{\partial^2 \phi}{\partial \delta^2} \quad (5.18) \]

This leads to

\[ \phi_{\infty, \infty}(x_0 + \delta) = 5.656 \times 10^{15} (-0.088 - 0.203\delta + 0.239\delta^2 - 0.316\delta^3) \quad (5.19) \]

Equation (5.19) includes terms that are linear, quadratic and cubic in \( \delta \). As a first attempt at modeling the nonlinearity of van der Waals forces, only the linear and cubic terms are considered. This is consistent with an assumption that the springs have the same properties in compression as in tension thus referring the force per unit length to be an odd function.

Table 5.1 The lattice sums of the graphite system given by Girifalco and Lad (4).

<table>
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<tr>
<th>( x )</th>
<th>( S_{\infty, \infty}^{12}(x) )</th>
<th>( S_{\infty, \infty}^{6}(x) )</th>
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<td>3.5</td>
<td>0.02428</td>
<td>3.50543 \times 10^{-6}</td>
</tr>
<tr>
<td>4.0</td>
<td>0.015158</td>
<td>9.2243 \times 10^{-7}</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0074934</td>
<td>9.3996 \times 10^{-8}</td>
</tr>
</tbody>
</table>
The van der Waals interaction pressure at any point ‘x’ between any two adjacent nanotubes should be a linear function of the jump deflection at that point. The resultant interaction pressure per unit axial length is proportional to the circumferential dimension, say the inner radius, of the adjacent nanotubes.

The interaction energy potential per unit axial length between any two adjacent nanotubes is \(2r_i g(\delta)\) where \(g(\delta)\) is a function of the interlayer spacing, \(\delta\) and \(r_i\) is the inner radius of the nanotube. Girifalco and Lad estimated \(g(\delta)\) by using the energy potential per unit area between two flat graphite sheets. So, the resultant interaction pressure from both sides between two adjacent nanotubes is given by,

\[
c_i = 2r_i \left. \frac{\partial^2 g}{\partial \delta^2} \right|_{\delta_{\text{opt}}} \tag{5.20}
\]

where \(t\) is the interlayer spacing of about 0.34 nm which is very close to the equilibrium interlayer spacing at which \(dg / d\delta = 0\). According to the data given by Girifalco and Lad, it is found that,

\[
\left. \frac{\partial^2 g}{\partial \delta^2} \right|_{\delta_{\text{opt}}} \sim \frac{366.67 \text{ erg/cm}^2}{0.16d^2}, \quad d = 0.142 \text{ nm}.
\tag{5.21}
\]

Then,

\[
c_i = \frac{366.67(2r_i) \text{ erg/cm}^2}{0.16d^2} \tag{5.22}
\]

where \(r_i\) denotes the inner radius of each pair of carbon nanotubes. The van der Waals forces between adjacent nanotubes are infinitely strong, if
The associated van der Waals forces and the interlayer radial displacements crucially affect the mechanical properties of carbon multiwalled nanotubes. The friction energy barrier between adjacent tubes in carbon multiwalled nanotubes is so low that the later can freely slide and rotate towards each other. Although the van der Waals forces resist the interlayer radial displacements, it is not clear whether the magnitude of the van der Waals interaction in multiwalled carbon nanotubes is strong enough to prevent any significant interlayer radial displacement.

The individual tubes of a multiwalled carbon nanotubes can deform independently with non zero interlayer radial displacements, while their individual deformations are coupled through the intertube van der Waals forces, due to different geometric radii and external conditions such as the different end conditions and surrounding conditions. The interlayer radial displacements and the non coincidence of the deflected tubes could affect the overall mechanical behavior of the multiwalled carbon nanotubes.

5.3 Example

Consider five identical elastic beams connected by non linear elastic springs. The outermost beams are rigid at one end and connected to the middle beam by elastic springs.
Table 5.2 The dimensional parameters of the five beams

<table>
<thead>
<tr>
<th>Dimensional Parameters</th>
<th>Beam 1</th>
<th>Beam 2</th>
<th>Beam 3</th>
<th>Beam 4</th>
<th>Beam 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density $\rho$ (kg/m$^3$)</td>
<td>1300</td>
<td>1300</td>
<td>1300</td>
<td>1300</td>
<td>1300</td>
</tr>
<tr>
<td>Area of the beams $A$ (m$^2$)</td>
<td>3.63.E-19</td>
<td>1.08.E-18</td>
<td>1.815.E-18</td>
<td>2.54.E-18</td>
<td>3.267.E-18</td>
</tr>
<tr>
<td>Moment of inertia $I$ (m$^4$)</td>
<td>0.01.E-36</td>
<td>0.16.E-36</td>
<td>0.68.E-36</td>
<td>1.836.E-36</td>
<td>3.87.E-36</td>
</tr>
</tbody>
</table>

The elastic spring constants between the springs $c_i$ are assumed as follows:

$$c = \begin{bmatrix} 7.73E +10 \\ 1.54564E +11 \\ 2.32E +11 \\ 3.1E +11 \\ 3.86E +11 \end{bmatrix}$$

and the non dimensional parameters are recorded as follows:

$$\phi = \begin{bmatrix} 1.653E +4 \\ 3.305E +4 \\ 4.96E +4 \\ 6.628E +4 \\ 8.253E +4 \end{bmatrix}$$
\[
\begin{bmatrix}
1 \\
2.975 \\
5 \\
6.997 \\
9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
16 \\
68 \\
183.6 \\
387 \\
\end{bmatrix}
\]

where \( \phi, \mu, \nu \) are defined in equation (3.6), (3.7) and (3.8) respectively.

Table 5.3 The linear natural frequencies for a fixed-fixed beam

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural frequency ( \omega_{n0} )</td>
<td>76.746</td>
<td>126.135</td>
<td>154.467</td>
<td>178.382</td>
<td>202.04</td>
</tr>
<tr>
<td>Intertube frequency ( \omega_{n1} )</td>
<td>125.419</td>
<td>198.293</td>
<td>262.983</td>
<td>341.321</td>
<td>414.849</td>
</tr>
<tr>
<td>Intertube frequency ( \omega_{n2} )</td>
<td>172.453</td>
<td>309.952</td>
<td>464.327</td>
<td>632.614</td>
<td>797.621</td>
</tr>
<tr>
<td>Intertube frequency ( \omega_{n3} )</td>
<td>236.559</td>
<td>481.56</td>
<td>748.398</td>
<td>1032</td>
<td>1314</td>
</tr>
<tr>
<td>Intertube frequency ( \omega_{n4} )</td>
<td>324.808</td>
<td>704.486</td>
<td>1109</td>
<td>1535</td>
<td>1960</td>
</tr>
</tbody>
</table>
A number of internal resonance conditions are identified. Internal resonances are considered when

\[ \pm \omega_{p,q} \pm \omega_{r,s} \pm \omega_{i,j} \leq 0.1 \]  

(5.24)

When this occurs a detuning parameter is defined such that

\[ \pm \omega_{p,q} \pm \omega_{r,s} = \omega_{i,j} + \varepsilon \sigma \]  

(5.25)

Here the internal resonances occur for different natural frequencies for the same \( \phi_k(x) \) as well as for distinct spatial modes that is, for modes corresponding to different values of index \( c \) but for either the same or different value of index \( l \).

The algebra is simplified by assuming

\[ A_{ki} = \frac{1}{2} d_{k,i} \left( T_1 \right) e^{i \phi_{k,i}(T_1)} \]  

(5.26)

where \( d_{k,i} \) is the amplitude and \( \phi_{k,i} \) is the phase

\[ \dot{A}_{ki} = \frac{1}{2} \left[ d_{k,i} \left( T_1 \right) e^{i \phi_{k,i}(T_1)} + i \dot{\phi}_{k,i} d_{k,i} \left( T_1 \right) e^{i \phi_{k,i}(T_1)} \right] \]  

(5.27)

and these should be substituted in the equation (xx), so the resulting \( n \) equations will have \( n \) unknowns to solve. These can be solved with help of fourth-order Runge-Kutta method. These solved unknowns can be substituted into the equation leading to a free response.

The internal resonances for this example are summarized in Table 5.3. In this table the case refers to the specific combination of frequencies comprising the resonant condition such that

Case A: \( \omega_{p,q} + \omega_{r,s} \)  

(5.28)

Case B: \( \omega_{p,q} - \omega_{r,s} \)  

(5.29)
Table 5.4 List of internal resonances

<table>
<thead>
<tr>
<th>$\omega_{p,q}$</th>
<th>$\omega_{r,s}$</th>
<th>Case</th>
<th>$\omega_{p,q} \pm \omega_{r,s}$</th>
<th>$\omega_{i,j}$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>2 1</td>
<td>A</td>
<td>202.165</td>
<td>198.293</td>
<td>38.7</td>
</tr>
<tr>
<td>1 1</td>
<td>2 3</td>
<td>A</td>
<td>339.729</td>
<td>341.321</td>
<td>16</td>
</tr>
<tr>
<td>1 1</td>
<td>2 4</td>
<td>A</td>
<td>418.067</td>
<td>414.849</td>
<td>32</td>
</tr>
<tr>
<td>1 1</td>
<td>2 1</td>
<td>A</td>
<td>202.165</td>
<td>202.04</td>
<td>12.5</td>
</tr>
<tr>
<td>1 3</td>
<td>2 3</td>
<td>A</td>
<td>417.45</td>
<td>414.849</td>
<td>26</td>
</tr>
<tr>
<td>1 3</td>
<td>3 2</td>
<td>A</td>
<td>464.419</td>
<td>464.327</td>
<td>0.92</td>
</tr>
<tr>
<td>1 5</td>
<td>2 1</td>
<td>B</td>
<td>76.721</td>
<td>76.746</td>
<td>1.25</td>
</tr>
<tr>
<td>1 5</td>
<td>1 1</td>
<td>B</td>
<td>125.294</td>
<td>125.419</td>
<td>12.5</td>
</tr>
<tr>
<td>2 1</td>
<td>2 2</td>
<td>A</td>
<td>323.712</td>
<td>324.808</td>
<td>109.6</td>
</tr>
<tr>
<td>2 1</td>
<td>2 4</td>
<td>A</td>
<td>466.74</td>
<td>464.327</td>
<td>241.3</td>
</tr>
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<td>2 2</td>
<td>2 3</td>
<td>A</td>
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</tr>
<tr>
<td>2 2</td>
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<td>B</td>
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<td>76.746</td>
<td>387.2</td>
</tr>
<tr>
<td>2 2</td>
<td>1 1</td>
<td>B</td>
<td>121.547</td>
<td>125.419</td>
<td>387.2</td>
</tr>
<tr>
<td>2 3</td>
<td>4 2</td>
<td>A</td>
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<td>262.983</td>
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<tr>
<td>2 4</td>
<td>2 3</td>
<td>B</td>
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<td>159.2</td>
</tr>
<tr>
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<td>B</td>
<td>338.103</td>
<td>341.321</td>
<td>32.18</td>
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</tbody>
</table>

51
Table 5.4 List of internal resonances (continued).

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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</tr>
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<tbody>
<tr>
<td>2 5</td>
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</tr>
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<td>151.866</td>
<td>154.467</td>
<td>260.1</td>
</tr>
<tr>
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<td>1 1</td>
<td>B</td>
<td>338.103</td>
<td>341.321</td>
<td>321.8</td>
</tr>
<tr>
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<td>A</td>
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</tr>
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<td>154.467</td>
<td>10.13</td>
</tr>
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<td>5 3</td>
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<td>797.6</td>
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</tr>
<tr>
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<td>3 5</td>
<td>A</td>
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<td>1109</td>
<td>142.7</td>
</tr>
<tr>
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<td>1 3</td>
<td>B</td>
<td>309.86</td>
<td>309.952</td>
<td>0.92</td>
</tr>
<tr>
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<td>B</td>
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<td>341.321</td>
<td>24.21</td>
</tr>
<tr>
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<td>2 2</td>
<td>B</td>
<td>266</td>
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<td>30.17</td>
</tr>
<tr>
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<td>2 4</td>
<td>B</td>
<td>123</td>
<td>125.419</td>
<td>24.19</td>
</tr>
<tr>
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<td>B</td>
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<td>154.467</td>
<td>9.2</td>
</tr>
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<td>2 3</td>
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<td>198.293</td>
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<td>B</td>
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<td>214.6</td>
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<tr>
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<td>309.952</td>
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</tr>
<tr>
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<td>A</td>
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<td>1032</td>
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<td>3 1</td>
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<td>309.952</td>
<td>8.52</td>
</tr>
<tr>
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<td>172.453</td>
<td>8.53</td>
</tr>
<tr>
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<td>B</td>
<td>266.838</td>
<td>262.983</td>
<td>385.5</td>
</tr>
</tbody>
</table>
Table 5.4 List of internal resonances (continued).

| 4 | 3 | 2 | 3 | B | 485.415 | 481.56 | 38.55 |
| 4 | 4 | 3 | 5 | B | 234.38  | 236.56 | 21.8  |
| 4 | 4 | 5 | 2 | B | 327.5   | 324.8  | 27    |
| 4 | 4 | 4 | 1 | B | 795.441 | 797.621| 218   |
| 4 | 4 | 5 | 1 | B | 707.192 | 704.486| 270.6 |
| 5 | 1 | 2 | 1 | B | 199.389 | 198.293| 109.6 |
| 5 | 1 | 5 | 2 | A | 1029.294| 1032   | 270.6 |
| 5 | 1 | 2 | 2 | B | 126.51  | 125.42 | 10.9  |
| 5 | 3 | 3 | 2 | B | 799     | 797.62 | 13.8  |
| 5 | 3 | 3 | 5 | B | 311.38  | 309.952| 14.28 |

Table 5.5 Definition of MATLAB variables

<table>
<thead>
<tr>
<th>MATLAB variables</th>
<th>Phase</th>
<th>MATLAB variables</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y1</td>
<td>$\phi_{11}$</td>
<td>Y26</td>
<td>$d_{11}$</td>
</tr>
<tr>
<td>Y2</td>
<td>$\phi_{12}$</td>
<td>Y27</td>
<td>$d_{12}$</td>
</tr>
<tr>
<td>Y3</td>
<td>$\phi_{13}$</td>
<td>Y28</td>
<td>$d_{13}$</td>
</tr>
<tr>
<td>Y4</td>
<td>$\phi_{14}$</td>
<td>Y29</td>
<td>$d_{14}$</td>
</tr>
<tr>
<td>Y5</td>
<td>$\phi_{15}$</td>
<td>Y30</td>
<td>$d_{15}$</td>
</tr>
</tbody>
</table>
Table 5.5 Definition of MATLAB variables (continued).

<table>
<thead>
<tr>
<th>Y6</th>
<th>(\phi_{21})</th>
<th>Y31</th>
<th>(d_{21})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y7</td>
<td>(\phi_{22})</td>
<td>Y32</td>
<td>(d_{22})</td>
</tr>
<tr>
<td>Y8</td>
<td>(\phi_{23})</td>
<td>Y33</td>
<td>(d_{23})</td>
</tr>
<tr>
<td>Y9</td>
<td>(\phi_{24})</td>
<td>Y34</td>
<td>(d_{24})</td>
</tr>
<tr>
<td>Y10</td>
<td>(\phi_{25})</td>
<td>Y35</td>
<td>(d_{25})</td>
</tr>
<tr>
<td>Y11</td>
<td>(\phi_{31})</td>
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<td>(d_{31})</td>
</tr>
<tr>
<td>Y12</td>
<td>(\phi_{32})</td>
<td>Y37</td>
<td>(d_{32})</td>
</tr>
<tr>
<td>Y13</td>
<td>(\phi_{33})</td>
<td>Y38</td>
<td>(d_{33})</td>
</tr>
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<td>(\phi_{34})</td>
<td>Y39</td>
<td>(d_{34})</td>
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<td>(\phi_{35})</td>
<td>Y40</td>
<td>(d_{35})</td>
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<td>(\phi_{41})</td>
<td>Y41</td>
<td>(d_{41})</td>
</tr>
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<td>Y17</td>
<td>(\phi_{42})</td>
<td>Y42</td>
<td>(d_{42})</td>
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<td>Y43</td>
<td>(d_{43})</td>
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<td>(\phi_{44})</td>
<td>Y44</td>
<td>(d_{44})</td>
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<td>Y20</td>
<td>(\phi_{45})</td>
<td>Y45</td>
<td>(d_{45})</td>
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<tr>
<td>Y21</td>
<td>(\phi_{51})</td>
<td>Y46</td>
<td>(d_{51})</td>
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<td>Y22</td>
<td>(\phi_{52})</td>
<td>Y47</td>
<td>(d_{52})</td>
</tr>
<tr>
<td>Y23</td>
<td>(\phi_{53})</td>
<td>Y48</td>
<td>(d_{53})</td>
</tr>
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</table>
Table 5.5 Definition of MATLAB variables (continued).

<table>
<thead>
<tr>
<th>Y24</th>
<th>$\phi_{34}$</th>
<th>Y49</th>
<th>$d_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y25</td>
<td>$\phi_{35}$</td>
<td>Y50</td>
<td>$d_{35}$</td>
</tr>
</tbody>
</table>
Set of first order differential equations

\[ \dot{\phi}_{11} = ((-6 \times 10^{-7} \times Y(10) \times Y(12) \times \cos(1.25t + Y(9) - Y(11) - Y(1))) + (7.511 \times 10^{-7} \times Y(14) \times Y(12) \times \cos(387.2t + Y(13) - Y(11) - Y(1))) + (1.544 \times 10^{-4} \times Y(18) \times Y(16) \times \cos(152.9t + Y(17) - Y(15) - Y(1))) + (2.5 \times 10^{-5} \times Y(20) \times Y(18) \times \cos(121.8t + Y(19) - Y(17) - Y(1)))) / Y(2) \]

(5.30)

\[ \dot{\phi}_{11} = (6 \times 10^{-7} \times Y(10) \times Y(12) \times \sin(1.25t + Y(9) - Y(11) - Y(1))) - (7.511 \times 10^{-7} \times Y(14) \times Y(12) \times \sin(387.2t + Y(13) - Y(11) - Y(1))) - (1.544 \times 10^{-4} \times Y(18) \times Y(16) \times \sin(152.9t + Y(17) - Y(15) - Y(1))) - (2.5 \times 10^{-5} \times Y(20) \times Y(18) \times \sin(121.8t + Y(19) - Y(17) - Y(1))) \]

(5.31)

\[ \dot{\phi}_{12} = 0 \]

(5.32)

\[ \dot{\phi}_{12} = 0 \]

(5.33)

\[ \dot{\phi}_{13} = ((-3.52 \times 10^{-4} \times Y(24) \times Y(6) \times \cos(10.13t + Y(23) - Y(5) - Y(5))) + (7.542 \times 10^{-5} \times Y(20) \times Y(16) \times \cos(260.1t + Y(19) - Y(15) - Y(5))) + (1 \times 10^{-3} \times Y(26) \times Y(24) \times \cos(9.2t + Y(25) - Y(23) - Y(5)))) / Y(6) \]

(5.34)

\[ \dot{\phi}_{13} = (3.52 \times 10^{-4} \times Y(24) \times Y(6) \times \sin(10.13t + Y(23) - Y(5) - Y(5))) - (7.542 \times 10^{-5} \times Y(20) \times Y(16) \times \sin(260.1t + Y(19) - Y(15) - Y(5))) - (1 \times 10^{-3} \times Y(26) \times Y(24) \times \sin(9.2t + Y(25) - Y(23) - Y(5))) \]

(5.35)

\[ \dot{\phi}_{14} = 0 \]

(5.36)

\[ \dot{\phi}_{14} = 0 \]

(5.37)

\[ \dot{\phi}_{15} = ((1.17 \times 10^{-4} \times Y(2) \times Y(12) \times \cos(12.5t + Y(1) + Y(11) - Y(9)) / Y(10) \]

(5.38)

\[ \dot{\phi}_{15} = ((-1.17 \times 10^{-4} \times Y(2) \times Y(12) \times \sin(12.5t + Y(1) + Y(11) - Y(9)) / Y(10) \]

(5.39)
\[ \dot{\phi}_{21} = ((8.6 \times 10^{-5} \cdot Y(26) \cdot Y(18) \cdot \cos(24.19t + Y(25) - Y(17) - Y(11))) + (8.6 \times 10^{-4} \cdot Y(42) \cdot Y(14) \cdot \cos(10.9t + Y(41) - Y(13) - Y(11))) - (1.185 \times 10^{-10} \cdot Y(2) \cdot Y(10) \cdot \cos(12.5t + Y(9) - Y(1) - Y(11))) + (7.5 \times 10^{-12} \cdot Y(14) \cdot Y(2) \cdot \cos(387.2t + Y(13) - Y(1) - Y(11)))) / Y(12) \]

\[ (5.40) \]

\[ \dot{\phi}_{21} = (8.6 \times 10^{-5} \cdot Y(26) \cdot Y(18) \cdot \sin(24.19t + Y(25) - Y(17) - Y(11))) - (8.6 \times 10^{-4} \cdot Y(42) \cdot Y(14) \cdot \sin(10.9t + Y(41) - Y(13) - Y(11))) + (1.185 \times 10^{-10} \cdot Y(2) \cdot Y(10) \cdot \sin(12.5t + Y(9) - Y(1) - Y(11))) - (7.5 \times 10^{-12} \cdot Y(14) \cdot Y(2) \cdot \sin(387.2t + Y(13) - Y(1) - Y(11))) \]

\[ (5.41) \]

\[ \dot{\phi}_{22} = ((3.347 \times 10^{-13} \cdot Y(2) \cdot Y(12) \cdot \cos(38.7t + Y(1) + Y(11) - Y(13))) + (9.185 \times 10^{-7} \cdot Y(42) \cdot Y(12) \cdot \cos(109.6t + Y(41) - Y(11) - Y(13))) + (2.9 \times 10^{-4} \cdot Y(26) \cdot Y(16) \cdot \cos(305.1t + Y(25) - Y(15) - Y(13)))) / Y(14) \]

\[ (5.42) \]

\[ \dot{\phi}_{22} = (3.347 \times 10^{-13} \cdot Y(2) \cdot Y(12) \cdot \sin(38.7t + Y(1) + Y(11) - Y(13))) - (9.185 \times 10^{-7} \cdot Y(42) \cdot Y(12) \cdot \sin(109.6t + Y(41) - Y(11) - Y(13))) - (2.9 \times 10^{-4} \cdot Y(26) \cdot Y(16) \cdot \sin(305.1t + Y(25) - Y(15) - Y(13))) \]

\[ (5.43) \]

\[ \dot{\phi}_{23} = ((2.75 \times 10^{-11} \cdot Y(18) \cdot Y(2) \cdot \cos(38.7t + Y(17) - Y(1) - Y(15))) - (1.2 \times 10^{-5} \cdot Y(20) \cdot Y(6) \cdot \cos(26t + Y(19) - Y(5) - Y(15))) + (1.6 \times 10^{-4} \cdot Y(26) \cdot Y(14) \cdot \cos(30.17t + Y(25) - Y(13) - Y(15))) + (9.6 \times 10^{-5} \cdot Y(36) \cdot Y(34) \cdot \cos(385.5t + Y(35) - Y(33) - Y(15)))) / Y(16) \]

\[ (5.44) \]

\[ \dot{\phi}_{23} = (2.75 \times 10^{-11} \cdot Y(18) \cdot Y(2) \cdot \sin(38.7t + Y(17) - Y(1) - Y(15))) + (1.2 \times 10^{-5} \cdot Y(20) \cdot Y(6) \cdot \sin(26t + Y(19) - Y(5) - Y(15)) + (1.6 \times 10^{-4} \cdot Y(26) \cdot Y(14) \cdot \sin(30.17t + Y(25) - Y(13) - Y(15))) - (9.6 \times 10^{-5} \cdot Y(36) \cdot Y(34) \cdot \sin(385.5t + Y(35) - Y(33) - Y(15))) \]

\[ (5.45) \]
\[ \dot{\phi}_{24} = (\frac{(-1.26 \times 10^{-12} \cdot Y(2) \cdot Y(16) \cdot \cos(16t + Y(1) + Y(15) - Y(17))) - (5.989 \times 10^{-3} \cdot Y(20) \cdot Y(2) \cdot \cos(32.18t + Y(19) - Y(1) - Y(17))) + (1.3 \times 10^{-3} \cdot Y(26) \cdot Y(12) \cdot \cos(24.21t + Y(25) - Y(11) - Y(17))) - (5.9898 \times 10^{-12} \cdot Y(20) \cdot Y(2) \cdot \cos(321.8t + Y(19) - Y(1) - Y(17)))}{Y(18)}) \] (5.46)

\[ \dot{\phi}_{24} = (\frac{1.26 \times 10^{-12} \cdot Y(2) \cdot Y(16) \cdot \sin(16t + Y(1) + Y(15) - Y(17))) + (5.989 \times 10^{-3} \cdot Y(20) \cdot Y(2) \cdot \sin(32.18t + Y(19) - Y(1) - Y(17))) - (1.3 \times 10^{-3} \cdot Y(26) \cdot Y(12) \cdot \sin(24.21t + Y(25) - Y(11) - Y(17))) + (5.9898 \times 10^{-12} \cdot Y(20) \cdot Y(2) \cdot \sin(321.8t + Y(19) - Y(1) - Y(17)))}{Y(20)}) \] (5.47)

\[ \dot{\phi}_{34} = (\frac{(6.15 \times 10^{-4} \cdot Y(2) \cdot Y(18) \cdot \cos(32t + Y(1) + Y(17) - Y(19))) - (1.11 \times 10^{-5} \cdot Y(6) \cdot Y(16) \cdot \cos(26t + Y(5) - Y(15) - Y(19)))}{Y(20)}) \] (5.48)

\[ \dot{\phi}_{34} = (\frac{-6.15 \times 10^{-4} \cdot Y(2) \cdot Y(18) \cdot \sin(32t + Y(1) + Y(17) - Y(19))) + (1.11 \times 10^{-5} \cdot Y(6) \cdot Y(16) \cdot \sin(26t + Y(5) - Y(15) - Y(19)))}{Y(20)}) \] (5.49)

\[ \dot{\phi}_{31} = (\frac{(9.5 \times 10^{-4} \cdot Y(34) \cdot Y(24) \cdot \cos(8.53t + Y(33) - Y(23) - Y(21)))}{Y(22)}) \] (5.50)

\[ \dot{\phi}_{31} = (\frac{-9.5 \times 10^{-4} \cdot Y(34) \cdot Y(24) \cdot \sin(8.53t + Y(33) - Y(23) - Y(21)))}{Y(22)}) \] (5.51)

\[ \dot{\phi}_{32} = (\frac{(-4.37 \times 10^{-5} \cdot Y(6) \cdot Y(26) \cdot \cos(0.92t + Y(25) - Y(5) - Y(23))) - (7.4 \times 10^{-5} \cdot Y(28) \cdot Y(42) \cdot \cos(21.52t + Y(27) - Y(41) - Y(23))) + (4.2 \times 10^{-4} \cdot Y(34) \cdot Y(22) \cdot \cos(8.52t + Y(33) - Y(21) - Y(23))) + (2.26 \times 10^{-6} \cdot Y(46) \cdot Y(30) \cdot \cos(14.3t + Y(45) - Y(29) - Y(23)))}{Y(24)}) \] (5.52)

\[ \dot{\phi}_{32} = (\frac{4.37 \times 10^{-5} \cdot Y(6) \cdot Y(26) \cdot \sin(0.92t + Y(25) - Y(5) - Y(23))) + (7.4 \times 10^{-5} \cdot Y(28) \cdot Y(42) \cdot \sin(21.52t + Y(27) - Y(41) - Y(23))) - (4.2 \times 10^{-4} \cdot Y(34) \cdot Y(22) \cdot \sin(8.52t + Y(33) - Y(21) - Y(23))) - (2.26 \times 10^{-6} \cdot Y(46) \cdot Y(30) \cdot \sin(14.3t + Y(45) - Y(29) - Y(23)))}{Y(24)}) \] (5.53)
\[
\dot{\phi}_{32} = ((-9.76 \times 10^{-5} \times Y(6) \times Y(24) \times \cos(0.92t + Y(5) - Y(23) - Y(25))) + (3 \times 10^{-8} \times \\
Y(12) \times Y(18) \times \cos(241.3t + Y(11) + Y(17) - Y(25))) + (9.745 \times 10^{-5} \times Y(14) \times Y(16) \times \cos(305.1t + \\
Y(13) + Y(15) - Y(25)))) / Y(26)
\]

(5.54)

\[
\dot{d}_{35} = (9.76 \times 10^{-5} \times Y(6) \times Y(24) \times \sin(0.92t + Y(5) - Y(23) - Y(25))) - (3 \times 10^{-8} \times \\
Y(12) \times Y(18) \times \sin(241.3t + Y(11) + Y(17) - Y(25))) - (9.745 \times 10^{-5} \times Y(14) \times Y(16) \times \sin(305.1t + \\
Y(13) + Y(15) - Y(25)))
\]

(5.55)

\[
\dot{\phi}_{34} = ((4 \times 10^{-4} \times Y(24) \times Y(42) \times \cos(21.46t + Y(23) + Y(41) - Y(27)))) / Y(28)
\]

(5.56)

\[
\dot{d}_{34} = (-4 \times 10^{-4} \times Y(24) \times Y(42) \times \sin(21.46t + Y(23) + Y(41) - Y(27)))
\]

(5.57)

\[
\dot{\phi}_{35} = ((5.6 \times 10^{-6} \times Y(46) \times Y(24) \times \cos(14t + Y(23) - Y(45) - Y(29))) + (5.6 \times 10^{-6} \times \\
Y(24) \times Y(46) \times \cos(13.8t + Y(45) - Y(23) - Y(29)) + (3.03 \times 10^{-5} \times Y(38) \times Y(32) \times \cos(218t + \\
Y(37) - Y(31) - Y(29)))) / Y(30)
\]

(5.58)

\[
\dot{d}_{35} = (-5.6 \times 10^{-6} \times Y(46) \times Y(24) \times \sin(14t + Y(23) - Y(45) - Y(29))) - (5.6 \times 10^{-6} \times \\
Y(24) \times Y(46) \times \sin(13.8t + Y(45) - Y(23) - Y(29)) - (3.03 \times 10^{-5} \times Y(38) \times Y(32) \times \sin(218t + \\
Y(37) - Y(31) - Y(29)))
\]

(5.59)

\[
\dot{\phi}_{41} = ((1.9 \times 10^{-5} \times Y(38) \times Y(30) \times \cos(21.8t + Y(37) - Y(29) - Y(31)))) / Y(32)
\]

(5.60)

\[
\dot{\phi}_{41} = (-1.9 \times 10^{-5} \times Y(38) \times Y(30) \times \sin(21.8t + Y(37) - Y(29) - Y(31)))
\]

(5.61)

\[
\dot{\phi}_{42} = ((6.24 \times 10^{-5} \times Y(36) \times Y(16) \times \cos(38.55t + Y(35) - Y(15) - Y(33))) + (9.95 \times 10^{-4} \times \\
Y(22) \times Y(24) \times \cos(84.5t + Y(21) + Y(23) - Y(33)))) / Y(34)
\]

(5.62)
\[
\dot{d}_{42} = (-6.24 \times 10^{-5} \cdot Y(36) \cdot Y(16) \cdot \sin(38.55t + Y(35) - Y(15) - Y(33))) - (9.95 \times 10^{-4} \cdot Y(22) \cdot Y(24) \cdot \sin(84.5t + Y(21) + Y(23) - Y(33)))
\] (5.63)

\[
\dot{\phi}_{43} = ((6.26 \times 10^{-5} \cdot Y(16) \cdot Y(34) \cdot \cos(38.55t + Y(15) + Y(33) - Y(35)))) / Y(36)
\] (5.64)

\[
\dot{d}_{43} = (-6.26 \times 10^{-5} \cdot Y(16) \cdot Y(34) \cdot \sin(38.55t + Y(15) + Y(33) - Y(35)))
\] (5.65)

\[
\dot{\phi}_{44} = ((1.2 \times 10^{-5} \cdot Y(30) \cdot Y(32) \cdot \cos(21.8t + Y(29) + Y(31) - Y(37))) + (8.56 \times 10^{-5} \cdot Y(42) \cdot Y(44) \cdot \cos(270.6t + Y(41) + Y(43) - Y(37)))) / Y(38)
\] (5.66)

\[
\dot{d}_{44} = (-1.2 \times 10^{-5} \cdot Y(30) \cdot Y(32) \cdot \sin(21.8t + Y(29) + Y(31) - Y(37))) - (8.56 \times 10^{-5} \cdot Y(42) \cdot Y(44) \cdot \sin(270.6t + Y(41) + Y(43) - Y(37)))
\] (5.67)

\[
\dot{\phi}_{45} = 0
\] (5.68)

\[
\dot{d}_{45} = 0
\] (5.69)

\[
\dot{\phi}_{51} = ((1.05 \times 10^{-4} \cdot Y(38) \cdot Y(44) \cdot \cos(27t + Y(37) - Y(43) - Y(41))) + (1.67 \times 10^{-7} \cdot Y(12) \cdot Y(14) \cdot \cos(109.6t + Y(11) + Y(13) - Y(41))) - (9 \times 10^{-6} \cdot Y(28) \cdot Y(24) \cdot \cos(214.6t + Y(27) - Y(23) - Y(41)))) / Y(42)
\] (5.70)

\[
\dot{d}_{51} = (-1.05 \times 10^{-4} \cdot Y(38) \cdot Y(44) \cdot \sin(27t + Y(37) - Y(43) - Y(41))) - (1.67 \times 10^{-7} \cdot Y(12) \cdot Y(14) \cdot \sin(109.6t + Y(11) + Y(13) - Y(41)) + (9 \times 10^{-6} \cdot Y(28) \cdot Y(24) \cdot \sin(214.6t + Y(27) - Y(23) - Y(41)))
\] (5.71)

\[
\dot{\phi}_{52} = ((1.2356 \times 10^{-4} \cdot Y(38) \cdot Y(42) \cdot \cos(270.6t + Y(37) - Y(41) - Y(43)))) / Y(44)
\] (5.72)

\[
\dot{d}_{52} = (-1.2356 \times 10^{-4} \cdot Y(38) \cdot Y(41) \cdot \sin(270.6t + Y(37) - Y(41) - Y(43)))
\] (5.73)
\[ \dot{\phi}_{33} = \frac{(1.868 \times 10^{-5} \times Y(24) \times Y(30) \times \cos(142.7t + Y(23) + Y(29) - Y(45)))}{Y(46)} \]  
(5.74)

\[ \dot{d}_{33} = (-1.868 \times 10^{-5} \times Y(24) \times Y(30) \times \sin(142.7t + Y(23) + Y(29) - Y(45))) \]  
(5.75)

\[ \dot{\phi}_{44} = 0 \]  
(5.76)

\[ \dot{d}_{44} = 0 \]  
(5.77)

\[ \dot{\phi}_{55} = 0 \]  
(5.78)

\[ \dot{d}_{55} = 0 \]  
(5.79)

The equations (5.30) to (5.79) represent the first order differential equations, which can be solved by numerical techniques. These equations are integrated using ode45 function of MATLAB which employs a fourth order Runge-Kutta method. Initial conditions of \( \phi_{y}(0) = 0 \) and \( d_{y}(0) = 0.01 \). The resulting plots are shown in the figure (5.1) – figure (5.38). We observe that the response has transient motion until 15000 after which it settles down to a steady state.
Figure 5.1 Response of amplitude $d_{11}$

Figure 5.2 Response of amplitude $d_{12}$
Figure 5.3 Response of amplitude $d_{13}$

Figure 5.4 Response of amplitude $d_{14}$
Figure 5.5 Response of amplitude $d_{15}$

Figure 5.6 Response of amplitude $d_{21}$
Figure 5.7 Response of amplitude $d_{22}$

Figure 5.8 Response of amplitude $d_{23}$
Figure 5.9 Response of amplitude $d_{24}$

Figure 5.10 Response of amplitude $d_{25}$
Figure 5.11 Response of amplitude $d_{31}$

Figure 5.12 Response of amplitude $d_{32}$
Figure 5.13 Response of amplitude $d_{33}$

Figure 5.14 Response of amplitude $d_{34}$
Figure 5.15 Response of amplitude $d_{35}$

Figure 5.16 Response of amplitude $d_{41}$
Figure 5.17 Response of amplitude $d_{42}$

Figure 5.18 Response of amplitude $d_{43}$
Figure 5.19 Response of amplitude $d_{44}$

Figure 5.20 Response of amplitude $d_{45}$
Figure 5.21 Response of amplitude $d_{51}$

Figure 5.22 Response of amplitude $d_{52}$
Figure 5.23 Response of amplitude $d_{53}$

Figure 5.24 Response of amplitude $d_{54}$
Figure 5.25 Response of amplitude $d_{55}$

Figure 5.26 Response of phase $\phi_{11}$
Figure 5.27 Response of phase $\phi_{21}$

Figure 5.28 Response of phase $\phi_{42}$
Substituting the response of amplitude and phase in equation (4.8) leads to the response of transverse displacement. The following initial conditions are chosen to derive the response of transverse displacement

\[ w_i(x, 0) = 0 \]
\[ w_2(x, 0) = 0 \]
\[ w_3(x, 0) = 0 \]
\[ w_4(x, 0) = 0 \]
\[ w_5(x, 0) = 0.25 x^2 (1 - x)^2 \]

\[ \frac{\partial w_i}{\partial t}(x, 0) = 0 \quad i = 1, 2...5 \] (5.80)

and the initial conditions for phase are zero.

Table 5.6 Initial conditions of amplitude

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<tr>
<th>(d_{ij})</th>
<th>(d_{11})</th>
<th>(d_{25})</th>
<th>(d_{32})</th>
<th>(d_{44})</th>
<th>(d_{55})</th>
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<td>1.455 \times 10^{-4}</td>
<td>1.61 \times 10^{-4}</td>
<td>1.569 \times 10^{-5}</td>
<td>1.691 \times 10^{-5}</td>
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<td>9.959 \times 10^{-3}</td>
<td>1.453 \times 10^{-4}</td>
<td>1.818 \times 10^{-5}</td>
<td>5.673 \times 10^{-5}</td>
<td>1.898 \times 10^{-6}</td>
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</tr>
<tr>
<td>0.011</td>
<td>1.61 \times 10^{-4}</td>
<td>1.818 \times 10^{-5}</td>
<td>5.673 \times 10^{-5}</td>
<td>1.898 \times 10^{-6}</td>
<td></td>
</tr>
<tr>
<td>1.275 \times 10^{-3}</td>
<td>1.453 \times 10^{-4}</td>
<td>1.818 \times 10^{-5}</td>
<td>5.673 \times 10^{-5}</td>
<td>1.898 \times 10^{-6}</td>
<td></td>
</tr>
<tr>
<td>3.974 \times 10^{-3}</td>
<td>1.818 \times 10^{-5}</td>
<td>5.673 \times 10^{-5}</td>
<td>1.898 \times 10^{-6}</td>
<td>5.884 \times 10^{-6}</td>
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<tr>
<td>2.486 \times 10^{-6}</td>
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<td>1.898 \times 10^{-6}</td>
<td>5.884 \times 10^{-6}</td>
<td>1.103 \times 10^{-6}</td>
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<tr>
<td>2.454 \times 10^{-6}</td>
<td>1.898 \times 10^{-6}</td>
<td>5.884 \times 10^{-6}</td>
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</tr>
</tbody>
</table>
Figure 5.29 Response of $w_1$ at $x = 0.25$

Figure 5.30 Response of $w_2$ at $x = 0.25$
Figure 5.31 Response of \( w_3 \) at \( x = 0.25 \)

Figure 5.32 Response of \( w_4 \) at \( x = 0.25 \)
Figure 5.33 Response of $w_5$ at $x = 0.25$

Figure 5.34 Response of $w_1$ at $x = 0.5$
Figure 5.35 Response of $w_2$ at $x = 0.5$

Figure 5.36 Response of $w_3$ at $x = 0.5$
Figure 5.37 Response of $w_4$ at $x = 0.5$

Figure 5.38 Response of $w_5$ at $x = 0.5$
A series of Euler-Bernoulli beams connected by flexible layers with quadratic nonlinearities is considered. A set of coupled partial differential equations are derived to model the vibrations of these beams.

The free response of the system with small displacements is developed using the method of multiple scales. The first order solution leads to the identification of system’s linear natural frequencies and mode shapes.

Modal analysis is used to determine the second order response. The second order equations are uncoupled using mode shape orthogonality of first order systems. Internal resonances at certain relations between natural frequencies at different modes are observed.

A system of five identical elastic beams connected by non linear elastic springs is considered. The outermost beams are rigid at one end and connected to the middle beam by elastic springs. Secular terms arising due to the internal resonances are eliminated.

The resulting 50 nonlinear first order ordinary differential equations for amplitude and phase are solved using ode45 Runge-Kutta routine in MATLAB.
The model is applied to the vibrations of multiwalled carbon nanotubes, where the flexible layers model van der Waals forces between atoms. Lennard-Jones potential function is used to derive a model of quadratic nonlinearity for small changes from an equilibrium position.

The work can be extended to identify the effects of combination resonances resulting from harmonic excitation. The effects of damping can be considered. The results can be used to refine predictions of free responses of carbon nanotubes.
REFERENCES


APPENDIX A

MATLAB PROGRAM WHICH SOLVES FOR AMPLITUDE AND PHASE

```
function dydt = nano(t,y)
  dydt = zeros(50,1);  % a column vector
  dydt(1) = ((-6*10^-7*y(10)*y(12)*cos(y(9)-y(11)-y(1)+12.5*t))+(1.544*10^-4*y(18)*y(16)*cos(y(17)-y(15)-y(1)+159.2*t))+(2.5*10^-5*y(18)*y(20)*cos(y(19)-y(17)-y(1)+121.8*t))+(7.511*10^-7*y(14)*y(12)*cos(y(13)-y(11)-y(1)+387.2*t)))/y(2);
  dydt(2) = (6*10^-7*y(10)*y(12)*sin(y(9)-y(11)-y(1)+12.5*t))-(1.544*10^-4*y(18)*y(16)*sin(y(17)-y(15)-y(1)+159.2*t))-(2.5*10^-5*y(18)*y(20)*sin(y(19)-y(17)-y(1)+121.8*t))-(7.511*10^-7*y(14)*y(12)*sin(y(13)-y(11)-y(1)+387.2*t));
  dydt(3) = 0;
  dydt(4) = 0;
  dydt(5) = ((-3.52*10^-4*y(24)*y(6)*cos(y(23)-y(5)-y(5)+10.13*t))+(7.542*10^-5*y(20)*y(16)*cos(y(19)-y(15)-y(5)+260.1*t))+(1*10^-3*y(24)*y(26)*cos(y(25)-y(23)-y(5)+9.2*t)))/y(6);
  dydt(6) = (3.52*10^-4*y(24)*y(6)*sin(y(23)-y(5)-y(5)+10.13*t))-(7.542*10^-5*y(20)*y(16)*sin(y(19)-y(15)-y(5)+260.1*t))-(1*10^-3*y(24)*y(26)*sin(y(25)-y(23)-y(5)+9.2*t));
  dydt(7) = 0;
  dydt(8) = 0;
  dydt(9) = (1.17*10^-13*y(2)*y(12)*cos(y(1)+y(11)-y(9)+12.5*t))/y(10);
  dydt(10) = (-1.17*10^-13*y(2)*y(12)*sin(y(1)+y(11)-y(9)+12.5*t));
  dydt(11) = ((8.6*10^-5*y(26)*y(18)*cos(y(25)-y(17)-y(11)+24.19*t))+(8.6*10^-4*y(42)*y(14)*cos(y(41)-y(13)-y(11)+10.9*t))-(1.185*10^-10*y(2)*y(10)*cos(y(9)+y(1)-y(11)+12.5*t))+(7.5*10^-12*y(14)*y(2)*sin(y(13)-y(1)-y(11)+387.2*t)))/y(12);
  dydt(12) = (-8.6*10^-5*y(26)*y(18)*sin(y(25)-y(17)-y(11)+24.19*t))-(8.6*10^-4*y(42)*y(14)*sin(y(41)-y(13)-y(11)+10.9*t))+(1.185*10^-10*y(2)*y(10)*sin(y(9)+y(1)-y(11)+12.5*t))-(7.5*10^-12*y(14)*y(2)*cos(y(13)-y(1)-y(11)+387.2*t));
  dydt(13) = ((3.347*10^-13*y(2)*y(12)*cos(y(1)+y(11)-y(13)+38.7*t))+(9.185*10^-7*y(42)*y(12)*cos(y(41)+y(11)-y(13)+109.6*t))+(2.9*10^-4*y(26)*y(16)*cos(y(25)-y(15)-y(13)+305.1*t)))/y(14);
```
\begin{align*}
\text{dydt}(14) &= (-3.347 \times 10^{-13} \cdot y(2) \cdot y(12) \cdot \sin(y(1)+y(11)-y(13)+38.7t)) - (9.185 \times 10^{-7} \cdot y(42) \cdot y(12) \cdot \sin(y(41)+y(11)-y(13)+109.6t)) - (2.9 \times 10^{-4} \cdot y(26) \cdot y(16) \cdot \sin(y(25)-y(15)+305.1t)); \\
\text{dydt}(15) &= ((2.75 \times 10^{-11} \cdot y(18) \cdot y(2) \cdot \cos(y(17)-y(1)-y(15)+15.92t)) - (1.2 \times 10^{-5} \cdot y(20) \cdot y(6) \cdot \cos(y(19)-y(5)-y(15)+26t)) + (1.6 \times 10^{-4} \cdot y(26) \cdot y(16) \cdot \cos(y(25)-y(15)+30.17t)) - (9.6 \times 10^{-5} \cdot y(36) \cdot y(34) \cdot \cos(y(35)-y(33)-y(15)+385.5t)))/(y(16)); \\
\text{dydt}(16) &= ((-2.75 \times 10^{-11} \cdot y(18) \cdot y(2) \cdot \sin(y(17)-y(1)-y(15)+15.92t)) + (1.2 \times 10^{-5} \cdot y(20) \cdot y(6) \cdot \sin(y(19)-y(5)-y(15)+26t)) - (1.6 \times 10^{-4} \cdot y(26) \cdot y(14) \cdot \sin(y(25)-y(13)-y(15)+30.17t)) - (9.6 \times 10^{-5} \cdot y(36) \cdot y(34) \cdot \sin(y(35)-y(33)-y(15)+385.5t)))/(y(16)); \\
\end{align*}
\[ \begin{align*}
\frac{dy_{dt}(33)}{dt} &= \frac{(6.24 \times 10^{-5} \cdot y(36) \cdot y(16) \cdot \cos(y(35)-y(15)-y(33)+38.55 \cdot t))+(9.95 \times 10^{-4} \cdot y(22) \cdot y(24) \cdot \cos(y(21)+y(23)-y(33)+84.5 \cdot t))}{y(34)}; \\
\frac{dy_{dt}(34)}{dt} &= \frac{(-6.24 \times 10^{-5} \cdot y(36) \cdot y(16) \cdot \sin(y(35)-y(15)-y(33)+38.55 \cdot t))-(9.95 \times 10^{-4} \cdot y(22) \cdot y(24) \cdot \sin(y(21)+y(23)-y(33)+84.5 \cdot t))}{y(34)}; \\
\frac{dy_{dt}(35)}{dt} &= \frac{(6.26 \times 10^{-5} \cdot y(16) \cdot y(34) \cdot \cos(y(15)+y(33)-y(35)+38.55 \cdot t))}{y(36)}; \\
\frac{dy_{dt}(36)}{dt} &= \frac{(-6.26 \times 10^{-5} \cdot y(16) \cdot y(34) \cdot \sin(y(15)+y(33)-y(35)+38.55 \cdot t))}{y(36)}; \\
\frac{dy_{dt}(37)}{dt} &= \frac{((1.2 \times 10^{-5} \cdot y(30) \cdot y(32) \cdot \cos(y(29)+y(31)-y(37)+21.8 \cdot t))+(8.56 \times 10^{-5} \cdot y(42) \cdot y(44) \cdot \cos(y(41)+y(43)-y(37)+270.6 \cdot t))}{y(38)}; \\
\frac{dy_{dt}(38)}{dt} &= \frac{(-1.2 \times 10^{-5} \cdot y(30) \cdot y(32) \cdot \sin(y(29)+y(31)-y(37)+21.8 \cdot t))-(8.56 \times 10^{-5} \cdot y(42) \cdot y(44) \cdot \sin(y(41)+y(43)-y(37)+270.6 \cdot t))}{y(38)}; \\
\frac{dy_{dt}(39)}{dt} &= 0; \\
\frac{dy_{dt}(40)}{dt} &= 0; \\
\frac{dy_{dt}(41)}{dt} &= \frac{((1.05 \times 10^{-4} \cdot y(38) \cdot y(44) \cdot \cos(y(37)-y(43)-y(41)+27 \cdot t))+(1.67 \times 10^{-7} \cdot y(12) \cdot y(14) \cdot \cos(y(11)+y(13)-y(41)+109.6 \cdot t))-(9 \times 10^{-6} \cdot y(28) \cdot y(24) \cdot \cos(y(27)-y(23)-y(41)+109.6 \cdot t))}{y(42)}; \\
\frac{dy_{dt}(42)}{dt} &= \frac{(-1.05 \times 10^{-4} \cdot y(38) \cdot y(44) \cdot \sin(y(37)-y(43)-y(41)+27 \cdot t))-(1.67 \times 10^{-7} \cdot y(12) \cdot y(14) \cdot \sin(y(11)+y(13)-y(41)+109.6 \cdot t))+(9 \times 10^{-6} \cdot y(28) \cdot y(24) \cdot \sin(y(27)-y(23)-y(41)+109.6 \cdot t))}{y(42)}; \\
\frac{dy_{dt}(43)}{dt} &= \frac{(1.2356 \times 10^{-4} \cdot y(38) \cdot y(42) \cdot \cos(y(37)-y(41)-y(43)+270.6 \cdot t))}{y(44)}; \\
\frac{dy_{dt}(44)}{dt} &= \frac{(-1.2356 \times 10^{-4} \cdot y(38) \cdot y(42) \cdot \sin(y(37)-y(40)-y(43)+270.6 \cdot t))}{y(44)}; \\
\frac{dy_{dt}(45)}{dt} &= \frac{(1.868 \times 10^{-5} \cdot y(24) \cdot y(30) \cdot \cos(y(23)+y(29)-y(45)+142.7 \cdot t))}{y(46)}; \\
\frac{dy_{dt}(46)}{dt} &= \frac{(-1.868 \times 10^{-5} \cdot y(24) \cdot y(30) \cdot \sin(y(23)+y(29)-y(45)+142.7 \cdot t))}{y(46)}; \\
\frac{dy_{dt}(47)}{dt} &= 0; \\
\frac{dy_{dt}(48)}{dt} &= 0; \\
\frac{dy_{dt}(49)}{dt} &= 0; \\
\frac{dy_{dt}(50)}{dt} &= 0;
\end{align*} \]
APPENDIX B

RUNGE-KUTTA METHOD

\[ y_0 = [0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01 0 0.01]; \]
\[ [t,y] = \text{ode45}(@nano,[0 25000],y0); \]
figure
plot(t,y(:,2));
title('Response for d_1_1')
xlabel('T_1')
ylabel('d_1_1')
figure
plot(t,y(:,4));
title('Response for d_1_2')
xlabel('T_1')
ylabel('d_1_2')
figure
plot(t,y(:,6));
title('Response for d_1_3')
xlabel('T_1')
ylabel('d_1_3')
figure
plot(t,y(:,8));
title('Response for d_1_4')
xlabel('T_1')
ylabel('d_1_4')
figure
plot(t,y(:,10));
title('Response for d_1_5')
xlabel('T_1')
ylabel('d_1_5')
figure
plot(t,y(:,12));
title('Response for d_2_1')
xlabel('T_1')
Ylabel('d_2_1')
figure
plot(t,y(:,14));
title('Response for d_2_2')
xlabel('T_1')
Ylabel('d_2_2')
figure
plot(t,y(:,16));
title('Response for d_2_3')
xlabel('T_1')
Ylabel('d_2_3')
figure
plot(t,y(:,18));
title('Response for d_2_4')
xlabel('T_1')
Ylabel('d_2_4')
figure
plot(t,y(:,20));
title('Response for d_2_5')
xlabel('T_1')
Ylabel('d_2_5')
figure
plot(t,y(:,22));
title('Response for d_3_1')
xlabel('T_1')
Ylabel('d_3_1')
figure
plot(t,y(:,24));
title('Response for d_3_2')
xlabel('T_1')
Ylabel('d_3_2')
figure
plot(t,y(:,26));
title('Response for d_3_3')
xlabel('T_1')
Ylabel('d_3_3')
figure
plot(t,y(:,28));
title('Response for d_3_4')
xlabel('T_1')
Ylabel('d_3_4')
figure
plot(t,y(:,30));
title('Response for d_3_5')
xlabel('T_1')
Ylabel('d_5_4')
figure
plot(t,y(:,50));
title('Response for d_5_5')
xlabel('T_1')
Ylabel('d_5_5')
APPENDIX C

MATLAB PROGRAM TO PLOT TRANSVERSE DISPLACEMENT

\[ y_0 = [0 \ 0.01 \ 0.959 \times 10^{-3} \ 0 \ 0.011 \ 0 \ 1.275 \times 10^{-3} \ 0 \ 3.974 \times 10^{-3} \ 0 \ 2.486 \times 10^{-6} \ 0 \ 2.454 \times 10^{-6} \ 0 \ 2.76 \times 10^{-6} \ 0 \ 3.111 \times 10^{-7} \ 0 \ 9.731 \times 10^{-7} \ 0 \ 1.455 \times 10^{-4} \ 0 \ 1.453 \times 10^{-4} \ 0 \ 1.61 \times 10^{-4} \ 0 \ 1.818 \times 10^{-5} \ 0 \ 5.673 \times 10^{-5} \ 0 \ 1.003 \times 10^{-6} \ 0 \ -9.995 \times 10^{-7} \ 0 \ -1.103 \times 10^{-6} \ 0 \ -1.243 \times 10^{-7} \ 0 \ -3.9 \times 10^{-7} \ 0 \ 1.582 \times 10^{-5} \ 0 \ 1.569 \times 10^{-5} \ 0 \ 1.691 \times 10^{-5} \ 0 \ 1.898 \times 10^{-6} \ 0 \ 5.884 \times 10^{-6}] \]

\[ [t, y] = \text{ode45}(@nano,[0 \ 400],y0); \]

figure
plot(t,y(:,2));
figure
plot(t,y(:,4));
figure
plot(t,y(:,6));
figure
plot(t,y(:,8));
figure
plot(t,y(:,10));
figure
plot(t,y(:,12));
figure
plot(t,y(:,14));
figure
plot(t,y(:,16));
figure
plot(t,y(:,18));
figure
plot(t,y(:,20));
figure
plot(t,y(:,22));
figure
plot(t,y(:,24));
figure
plot(t,y(:,26));
figure
plot(t,y(:,28));
figure
plot(t,y(:,30));
figure
plot(t,y(:,32));
figure
plot(t,y(:,34));
figure
plot(t,y(:,36));
figure
plot(t,y(:,38));
figure
plot(t,y(:,40));
figure
plot(t,y(:,42));
figure
plot(t,y(:,44));
figure
plot(t,y(:,46));
figure
plot(t,y(:,48));
figure
plot(t,y(:,50));
a=[1 1.026 1.22 0.317 0.805; 0.999 0.407 0.182 0.317 0.805; -1 -1.52 0.245 0.637 0.162; 
-1 0.315 0.553 0.143 0.365; -0.99 0.217 0.864 0.224 0.57];
a(:,:,2)=[1 1 0.63 0.27; 1 0.999 0.99 0.626 0.268; 1 0.998 0.995 0.623 0.267; 1 0.997 
0.986 0.618 0.265; 0.999 0.99 0.944 0.592 0.254];
a(:,:,3)=[1 1 0.39 0.6436; 1 0.999 1.01 0.386 0.637; 1 0.999 1.035 0.385 0.635; 1 0.996 1 
0.382 0.63; 0.999 0.991 0.96 0.367 0.605];
a(:,:,4)=[1 1 1.06 0.222 0.148; 1 1 1.058 0.222 0.148; 0.999 0.997 1.04 0.218 0.1456; 1 
0.998 1.03 0.2178 0.1452; 1 0.992 0.998 0.209 0.139];
a(:,:,5)=[0.999 1 1.132 0.128 0.399; 0.999 0.986 1.109 0.125 0.391; 1 0.999 1.107 0.125 
0.39; 1 0.997 1.1 0.124 0.389; 1 0.992 1.069 0.12 0.377];
omega=[76.746 126.135 154.467 178.382 202.04; 125.419 198.293 262.983 341.321 
414.849; 172.453 309.952 464.327 632.614 797.621; 236.559 481.56 748.398 1032 
1314; 324.808 704.486 1109 1535 1960];
q=length(t);
b1=[0.863 1.445 1.372 0.569 -0.528];
b2=[1.588 8.759*10^-4 -1.406 2.002*10^-3 1.415];
b3=[0.863 -1.445 1.368 -0.573 -0.527];
for p=1:q
time=t(p)/0.1;
for k=1:5
w25(k,p)=0;
w5(k,p) = 0;
w75(k,p) = 0;
for i = 1:5
  for j = 1:5;
    s = 10*(i-1)+2*j;
    u = s-1;
    w25(k,p) = a(i,j,k)*y(p,s)*b1(i)*cos(omega(i,j)*time+y(p,u)*0.1*time)+w25(k,p);
    w5(k,p) = a(i,j,k)*y(p,s)*b2(i)*cos(omega(i,j)*time+y(p,u)*0.1*time)+w5(k,p);
    w75(k,p) = a(i,j,k)*y(p,s)*b3(i)*cos(omega(i,j)*time+y(p,u)*0.1*time)+w75(k,p);
  end
end
end
figure
plot(t,w25(1,:));
title('Response for w_1(0.25,t)')
xlabel('t')
Ylabel('w_1(0.25)')
figure
plot(t,w25(2,:));
title('Response for w_2(0.25,t)')
xlabel('t')
Ylabel('w_2(0.25)')
figure
plot(t,w25(3,:));
title('Response for w_3(0.25,t)')
xlabel('t')
Ylabel('w_3(0.25)')
figure
plot(t,w25(4,:));
title('Response for w_4(0.25,t)')
xlabel('t')
Ylabel('w_4(0.25)')
figure
plot(t,w25(5,:));
title('Response for w_5(0.25,t)')
xlabel('t')
Ylabel('w_5(0.25)')
figure
plot(t,w5(1,:));
title('Response for w_1(0.5,t)')
xlabel('t')
Ylabel('w_1(0.5)')
figure
plot(t,w5(2,:));
title('Response for w_2(0.5,t)')
xlabel('t')
ylabel('w_2(0.5)')
figure
plot(t,w5(3,:));
title('Response for w_3(0.5,t)')
xlabel('t')
ylabel('w_3(0.5)')
figure
plot(t,w5(4,:));
title('Response for w_4(0.5,t)')
xlabel('t')
ylabel('w_4(0.5)')
figure
plot(t,w5(5,:));
title('Response for w_5(0.5,t)')
xlabel('t')
ylabel('w_5(0.5)')