STOCHASTIC ANALYSIS AND OPTIMIZATION OF STRUCTURES

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ABSTRACT

In structural engineering problems, uncertainty is inherent in the load, strength and material property. The resulting stochastic problem can be solved numerically using the computationally intensive Monte Carlo technique. The stochastic finite element method is an alternative approach. This method is based on the perturbation technique. Uncertainties are considered as random variables with a relatively modest fluctuation about the mean. The present study develops the perturbation formulation as the primary stochastic analysis tool. The formulation is analytically elegant and numerically inexpensive. The stochastic analyzer is integrated next into the design optimization testbed CometBoards of NASA Glenn Research Center. The design tool in the stochastic domain was also extended to obtain a robust formulation that can minimize the variation of the objective function.

The stochastic analysis utilizes both force and displacement formulations. The force formulation in the literature is referred to as the integrated force method (IFM). Its dual or the dual integrated force method (IFMD) became the stiffness formulation. The first- and second-order perturbation techniques were applied to the governing formulae of force and displacement methods to obtain closed form expressions for the mean and standard deviation of response parameters consisting of internal force, displacement and member stress. Stochastic sensitivity analysis was formulated for selected response
variables. The analytical methods also included Neumann expansion with Monte Carlo simulation as well as a variational energy formulation and simplification and reduction on stochastic calculations. Formulas of the stochastic analysis were programmed in Maple V software as well as in the FORTRAN language. Solutions were obtained for a set of examples, which were verified via Monte Carlo simulations. The IFM/IFMD perturbation methods yield response very efficiently for modest fluctuation in random variables. The difference between the first- and second-order perturbations methods was small. The stochastic sensitivity analysis also exhibited a similar trend. The mean value and standard deviation of response are in a good agreement with Monte Carlo simulation obtained for both IFM/IFMD and the regular stiffness method.

Optimum designs were generated for the same set of examples for mechanical, thermal and initial deformation loads. The deterministic and stochastic solutions as well as robust designs were compared. The stochastic design solution matched the deterministic results for a fifty percent probability of success \( (p = 0.5) \); for other success level the mean value of the objective function increased or decreased with increasing or decreasing the probability of success. The standard deviation of the objective function followed a pattern that was similar to its mean value. The difference between the first- and second-order perturbations was small for both the mean value and standard deviation of objective function or weight of the structure.

In summary, this study investigated the probabilistic analytical methods and stochastic optimization for structures. A set of illustrative examples were solved for stochastic analysis, sensitivity analysis and optimization. Structural response and design are influenced by the primitive random variables.
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CHAPTER I
INTRODUCTION

Traditionally, an engineering problem is solved on the basis of a deterministic model with well-defined parameters. However, the parameters of such problem contain uncertainties, for example, in material properties, external loads, structural geometry as well as other measurement inaccuracies. Uncertainty may be modeled using the probability distribution functions. Response can be determined using the theory of probability and stochastic mechanics.

In general, direct Monte Carlo simulation (MCS), which consists of solving deterministic governing equations for numerous, probabilistically selected realizations of random variables, is certainly a powerful numerical approach to the kinds of problems. However, its simulation procedures are computationally repetitive and costly, even though it is easily applicable to both linear and nonlinear systems. Hence, the non-statistical approaches, such as numerical integration, second-moment analysis and stochastic finite element methods, are available. The objective of stochastic mechanics is to develop approximate expressions for response uncertainty in primitive variables. This approach is based on the perturbation technique. The response is expanded about the mean value of primitive variables.
Stochastic finite element method (SFEM) employs the mean-based second-moment analysis to get formulations in combination with the first- or second-order perturbation technique. Based on the effective tool of the finite element method, SFEMs are applicable for performing analysis of structures with uncertain properties. Only the first two probabilistic moments of random variables are required as input parameters. SFEMs also are simple in concept. Its computer implementation is straightforward. The flexibility accommodates efficient numerical techniques, but systems need the relatively small fluctuation in random variables and the unique and smooth solutions.

It should be noted that almost all the investigations in SFEM appear to use the displacement method. Although the displacement method has become prevalent in structural analysis, its shortcomings have been observed in analysis of certain problems, such as the material treatment, design optimization accuracy of stress and the computational burden. An alternative force formulation, termed in the Integrated Force Method (IFM), has been developed in recent years for the analysis of structural mechanics. This formulation was proposed by Patnaik [1-11] for the analysis of discrete and continuous systems.

The Integrated Force Method, force method of analysis, integrates the system equilibrium equations and the global compatibility conditions in a fashion paralleling approach in continuum mechanics. It is demonstrated in studies that IFM has several advantages over the displacement method.

1) IFM is able to provide more accurate stress results than the displacement method.

2) IFM equations for finite element discrete analysis form a well-conditional system.
3) In discrete analysis solutions, the convergence to correct solutions in IFM is faster than that in the stiffness method. IFM require less computation time for certain problems than does the displacement method.

4) Initial deformation problems are more elegantly treated in IFM than in stiffness method.

With further study, IFM can become a robust and versatile analysis formulation and an equally viable alternative to the present popular displacement method. As mentioned above, it is more significant and necessary to develop the stochastic analysis for IFM. In this study, we attempt to solve for formulations of the probabilistic structural analysis for the Integrated Force Method/Dual Integrated Force Method by using the first- and second-order perturbation technique.

The sensitivity analysis of structural systems to variations in their parameters is one of the ways to evaluate the performance of structures. In general, sensitivity analysis of structural system involves computation of the partial derivatives of some response function with respect to the design parameters. It is important for system optimization, parameter identification, reliability assessment etc. in structural analysis. However, the conventional sensitivity analysis of structures is based on the assumptions of complete determinacy of structural parameter. In reality, the occurrence of uncertainty associated with the system parameters is inevitable. Hence, there is a necessity to estimate the effect of uncertainty in stochastic sensitivity derivatives with respect to random design variables.

The stochastic structural sensitivity analysis is concerned with the change of stochastic structural response due to variations in random design parameters. There are
several methods for calculating the sensitivity of structural response to change in design parameters, including analytical, numerical and semi-analytical methods.

In this study, the semi-analytical method is employed in the stochastic sensitivity analysis for Integrated Force Method and Dual Integrated Force Method (IFM/IFMD), since this method based on the perturbation technique is as easy to implement as numerical approach and as efficient as analytical calculation.

Optimization techniques in the design of engineering structures have been applied to versatile problems for improving the structural performance and reducing manufacture cost. However, these optimization approaches mostly make use of the deterministic parameters with reference to nominal values of design variables and other structural parameters. In practice, these parameters, such as elasticity modules, yield stresses, allowable stresses, moment of inertia area, external loadings, thermal temperature and so on, may be subject to random fluctuations or uncertainties due to system design and manufacturing process. At the optimum, the variation of design variables can cause constraints uncertainty and also give rise to a change of performance. Since the deterministic approaches of optimization neglect the effects from uncertainties of design variables, the design of deterministic optimization may not achieve the desired optimal goal or may become infeasible. Therefore, it is reasonable to explore the effect of randomness in the design optimization.

The stochastic optimization with random design variables is the process of maximizing or minimizing a desired objective function while satisfying the probabilistic constraints. With the Monte Carlo simulation or stochastic approximation method, the
stochastic optimization can incorporate the uncertain design variables into the procedure to obtain the feasible objective function under given probabilistic constraints.

In this framework, the robust design of structures with random variables is also introduced in the present study. The objective of robust design is to find the minimization of the mean, also the variation in objective function while maintaining feasibility with probabilistic constraints. In particular, the structural performance is required to be less sensitive to the variations on design variables within a feasible region. The robust structural design can reduce the variability of structural performance caused by regular fluctuations. Thus, the robust design has greatly attracted researchers’ attention. In the present study, the robust design of structures is formulated as a two-criterion optimization problem, in which both the expected value and the standard derivation of the weight objective function are minimized by optimization techniques with perturbation.

It is obvious that many theories and algorithms in structural optimization for a minimum weight condition have matured over the past decades. This is evidenced by the emergence of many design codes: CometBoards [12-18], Astros [19], Genesis [20], IDESIGN [21], ANSYS [22] and such. CometBoards, developed by NASA Lewis Research Center, is a general-purpose optimization tool for multidisciplinary applications. The modular organization of CometBoards includes several analyzers and state-of-the-art optimization algorithms along with their cascading strategy. The code allows quick integration of new analyzers and optimizers.

Recently, CometBoards is being extended into the probabilistic design. In order to formulate the probabilistic structural analysis code, a treatment of probabilistic nonlinear problems with deterministic computational techniques is employed to take full advantage
of the mathematical properties of linear operators by using the perturbation methods. Then the stochastic analyzer is integrated into CometBoards for design optimization. This study is to investigate the stochastic optimization in CometBoards.

In this study, a full set of stochastic analysis formulas for IFM/IFMD has been completed by using the first- and second-order moment perturbation methods. The sensitivity analysis of stochastic responses has also been studied. Neumann expansion technique for stochastic analysis using IFM/IFMD has been also described. Furthermore, the variational energy formulation for stochastic analysis in IFM is provided. Those closed-form expressions for stochastic analysis have been programmed in Maple V and FORTRAN code and their analysis solutions for simple structures have been verified via Monte Carlo simulations. After the constraints are expressed in random variables associated with probability distributions, the stochastic optimization with the expected objective has been performed on CometBoards. To reduce the influence of the variation on the performance, the robust design minimizing both mean and variation of objective has also been provided. 15 numerical examples of structures are solved to illustrate the stochastic analysis and optimization. Some results in stochastic analysis are compared with Monte Carlo simulation, Neumann expansion technique and ANSYS software [22].

The subject matter of the proposal is presented in subsequent six chapters. Chapter 2 is devoted to a review of literature on stochastic analysis, stochastic sensitivity analysis and stochastic optimization of structures. In Chapter 3, the basic theory of the primal/dual integrated force methods and the variational formulas of IFM are presented. Based on the perturbation technique, the first- and second-order approximation formulations of the stochastic analysis for IFM/IFMD are developed. Meanwhile a
simplification of calculation is considered. Application of the Neumann expansion with Monte Carlo simulation for stochastic analysis of IFM/IFMD is also explored. As an evaluating tool of structural analysis, the stochastic sensitivity analysis of IFM/IFMD is studied. Based on the potential energy variational principle, the stochastic variational principles of IFM has been theoretically developed. In Chapter 4, the stochastic formulations of objective function and constraints are derived. The Sequential Quadratic Programming algorithm in CometBoards is briefly introduced. Moreover, the formulations of the stochastic optimization are provided for the expected objective function and the robust design. Chapter 5 includes the brief introduction of direct Monte Carlo simulation and Latin hypercube sampling, 15 numerical examples of stochastic analysis and design optimization of several types of structures are given to illustrate an application of these methods. Some results in stochastic analysis and stochastic sensitivity analysis are compared with Monte Carlo simulation, Neumann Expansion technique and ANSYS software. Conclusion and discussion are the subject matter of the last chapter.
CHAPTER II
LITERATURE REVIEW

In recent decades, the stochastic analysis of structures, involving spatially random material, geometry and external loads, has attracted significant interest of many researchers. The stochastic theory is well established and has been extensively used for predicating probabilistic response and reliability of structural and mechanical systems. In this chapter, attention is focused on the review of probabilistic analysis, stochastic sensitivity analysis and stochastic optimization of structures in the linear static domain.

2.1 Probabilistic analysis of structures

Probabilistic method in static linear mechanics can be mainly classified into two categories:

1) Methods using a statistical approach.

2) Methods using a non-statistical approach

In the first class, Monte Carlo simulation (MCS) is the most prevalent statistical approach. It has long been used for the solution of probabilistic and statistical problem in many fields [23, 24]. Of course, this method has also been applied to the probabilistic structural analysis. Direct Monte Carlo simulation and variance reduction techniques such as the importance sampling method, stratified sampling method, Latin hypercube
sampling method, the response surface method and so on, have been developed during the past three decades. An applied and comparative discussion of these techniques can be found in reference [25]. Ayyub and Lai [26], for example, used Latin hypercube sampling to assess structural reliability. Melchers [27] improved the importance sampling to calculate the structural system reliability. Within the framework of Monte Carlo methods, Yamazaki et al. [28] developed the finite element solution for the response variability by using the Neumann expansion technique. Lima and Ebecken [29] recently compared the behavior of three methodologies: direct Monte Carlo simulation, stochastic finite element method and fuzzy finite element method. They came to a conclusion that the classic Monte Carlo method is a well-known and general methodology, because of its versatility and algorithmic straightforwardness. Simulations, however, are usually prohibitively expensive. The SFEM is accurate and efficient under relatively small fluctuation in random variables. The fuzzy finite element method can lead to expeditions and qualitative results.

Although MCS can solve virtually any probabilistic structural mechanics problem, the simulations’ cost makes it inapplicable. For this reason many researchers have focused their interests on the non-statistical approach.

In non-statistical approaches, there are mainly three methods: numerical integration [30], second-moment analysis [31] and stochastic finite element method [32]. In particular, the perturbation method has been used extensively in developing the stochastic finite element method, because of its simplicity, efficiency and versatility. Based on Monte Carlo simulation, the SFEM was indicated by Astill et al. [33]. Cambou [34] appears to proposed first the first-order perturbation method for the finite element
solution of linear static problems with loading and system stochasticity; the idea of second-moment analysis was put forth earlier by Cornell [35]. In the early 1980s, the method was systematically developed by Hisada and Nakagiri [36, 37] for static and eigenvalue problems with application to the estimation of structural reliability. The perturbation method in SFEM has also been adopted by Handada and Anderson [38] for static problems of beam and frame structures, by Ishii and Suzuki [39] for slope stability reliability analysis and by Righetti and Harrop-Williams [40] for static stress analysis in soils. Liu et al. introduced a new implementation scheme for the perturbation based on finite element method. The perturbation expansion was carried on at a different stage of the finite element assembly process. Liu et al. also applied this approach to analyzing the linear static problems [41], nonlinear dynamics problems [42], and inelastic problems [43]. Based on Fourier analysis, Liu et al. [44, 45] solved the numerical instability resulting from the secular terms. The perturbation methods have provided efficient and accurate results for small random fluctuation in the random variables. Some extensive surveys on the applications of perturbation methods in developing the SFEM have been given by Benaroya and Rehak [46], Eleishakoff and Ren [47], Schuëller [48].

Here, the perturbation method and the Neumann Expansion method are reviewed in the linear static problems.

In static problems, the equilibrium equations of the finite element discretization of the structure can be written as

\[ KU = F \]  \hspace{1cm} (2.1)

in which \( K \) is the random global stiffness matrix, \( U \) is the random vector of unknown nodal displacements and \( F \) is the random vector of equivalent nodal forces. Assume that
the stiffness matrix $K$ involves a set of non-dimensional random variables $a = [a_1, a_2, ..., a_n]^T$, each $a_i$ is small ($a_i << 1$) and has zero mean. By means of Taylor series, $K$ can be expanded in the following form,

$$K = K^0 + \sum_{i=1}^{n} K_i a_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_{ij} a_i a_j + ...$$  \hspace{1cm} (2.2)

in which $K^0$ is the stiffness matrix at $a = \{0\}$, and $K_i$ and $K_{ij}$ are partial derivatives of $K$ defined as follows:

$$K_i = \frac{\partial K}{\partial a_i} \bigg|_{a = \{0\}} \hspace{1cm} (2.3a)$$

$$K_{ij} = \frac{\partial^2 K}{\partial a_i \partial a_j} \bigg|_{a = \{0\}} \hspace{1cm} (2.3b)$$

Similarly, the force vector $F$ and the displacement vector $U$ are also expanded as

$$F = F^0 + \sum_{i=1}^{n} F_i a_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} a_i a_j + ...$$  \hspace{1cm} (2.4)

$$U = U^0 + \sum_{i=1}^{n} U_i a_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij} a_i a_j + ...$$  \hspace{1cm} (2.5)

Substituting equations (2.2), (2.4) and (2.5) into equation (2.1) and equating equal order terms, we get the following first-three equations corresponding to zero, first and second order perturbations.

Zero order: \hspace{1cm} $K^0 U^0 = F^0$

First order: \hspace{1cm} $K_i U^0 + K^0 U_i^f = F_i^f$  \hspace{1cm} (2.6)

Second order: \hspace{1cm} $K_{ij} U^0 + K_i U_j^f + K_j U_i^f + K^0 U_{ij}^f = F_{ij}^f$

The solution of Eq.(2.6) can be written as
The first-order approximation for the displacement is obtained by truncating the right-hand side of Eq.(2.5) after the second term as

\[ U = U^0 + \sum_{i=1}^{n} U_i^0 a_i \]  

(2.8)

The SFEM based on the first-order perturbation will give rise to the following mean and covariance of the response \( U \) as

\[
E'[U] = U^0
\]

\[
\text{Cov}'(U,U^T) = E[(U - E'[U])(U - E'[U])^T] = \sum_{i=1}^{n} \sum_{j=1}^{n} U_i'^0 (U_j'^0)^T E[a,a_j]
\]

(2.9)

where \( E[a,a_j] \) is determined analytically from the autocorrelation function of the underlying stochastic field of \( a \).

Analogous to the first-order approximation, the second-order approximation for the displacement is also obtained by truncating the right-hand side of Eq.(2.5) after the third term as

\[
U = U^0 + \sum_{i=1}^{n} U_i^0 a_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij}'' a_i a_j
\]

(2.10)

The SFEM based on second-order perturbation gives

\[
E''[U] = U^0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij}'' E[a,a_j]
\]
\[ \text{Cov}^H(U,U^T) = \text{Cov}^I(U,U^T) \]
\[ + \frac{1}{4} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} \sum_{l=1}^{n_4} U_{ij}^H(U_{ij}^H)^T \left( E[a_1,a_1]E[a_2,a_1] + E[a_1,a_2]E[a_1,a_1] \right) \] (2.11)

In the process of deriving \( \text{Cov}^I(U,U^T) \), the \( a \)s are assumed to be Gaussian procedure. If they are not Gaussian, one must evaluate their third-and fourth-order moments accordingly. The expected values and covariance matrices of strain and stress are also evaluated analytically in a manner similar to that for the displacement.

It is seen that the second-order SFEM constitutes much more complex computation than the first-order SFEM. Some exact solutions are obtained to conduct the comparison between SFEM and the exact solutions in Reference [49].

It is noted that the analysis of multidimensional and multivariate stochastic fields was carried out earlier by Shinazuka [50] and his associates. Under the Neumann series, Shinazuka [51] investigated the deterministic static problem of an axially loaded rod with a random Young’s modulus. Yamazaki et. al. [28] took advantage of the Neumann expansion technique in deriving the finite element solution for the response variability within the framework of Monte Carlo methods and compared accuracy, convergence and computational efficiency with the perturbation and Monte Carlo solutions.

Consider again a finite element discretization and the corresponding matrix Eq.(2.1). Under the assumption that \( K \) contains spatial variable parameters, the matrix \( K \) can be represented as

\[ K = K_0 + \Delta K \] (2.12)

where \( K_0 \) denotes a deterministic stiffness matrix in which the spatially variable parameters are replaced by their mean values. \( \Delta K \) consists of components representing
the random fluctuations of the corresponding components in $K$, i.e. $\Delta K = K - K_0$. The deterministic solution of displacement $U_0$ which corresponding to $K_0$ can be obtained as

$$U_0 = (K_0)^{-1} F$$  \hspace{1cm} (2.13)

By using the Neumann expansion, the approximate inverse stiffness matrix $K^{-1}$ can be expressed in the following form:

$$K^{-1} = (K_0 + \Delta K)^{-1} = (I - P + P^2 - P^3 + ...)K_0^{-1}$$  \hspace{1cm} (2.14)

where $P = (K_0)^{-1} \Delta K$. Substituting expression Eq.(2.14) with Eq.(2.1) and using Eq.(2.13), the solution vector $U$ takes the form:

$$U = U_0 - PU_0 + P^2U_0 - P^3U_0 + ...$$  \hspace{1cm} (2.15)

Hence, it is a series of successive higher order corrections:

$$U = U_0 - U_1 + U_2 - U_3 + ...$$  \hspace{1cm} (2.16)

The above series solution is equivalent to the recursive

$$U_0 = (K_0)^{-1} F$$

$$K_0 U_i = \Delta K U_{i-1} \hspace{1cm} i = 1, 2, ...$$  \hspace{1cm} (2.17)

The expansion series in Eq.(2.15) may be terminated after a few terms when convergence of the series is obtained by using the following criterion:

$$\frac{\left\|U_i\right\|_2}{\sum_{k=0}^{i} (-1)^k \left\|U_k\right\|_2} \leq \delta_{err}$$  \hspace{1cm} (2.18)

where $\delta_{err}$ is the allowable error to be specified for convergence, $\left\|\cdot\right\|_2$ is the vector norm (length) defined by
The Neumann expansion method is especially suitable when the Monte Carlo simulation is used for the spatial variation of material properties. It should be underlined that the matrix inversion is required only once for the mean-valued stiffness matrix. Therefore, the computational time and costs may be reduced considerably.

2.2 Stochastic sensitivity analysis

In probabilistic structural analysis, system optimization, identification and reliability assessment, stochastic sensitivities provide important information for assessing the effect of uncertainties by predicting the change in response and optimizing a system. There has been a great interest in developing various methods for computing response sensitivity of structure [52-54]. The literature, however, on the stochastic sensitivity analysis is limited. Hien and Kleiber [55] formulated the stochastic design sensitivity problems of structural static in a effective way, by using the perturbation approach and adjoint variables method. Numerical algorithms are developed and turn out to be readily adaptable to existing finite element codes. The structural design sensitivity and the stochastic finite elements are similar in terms of the methodology and computer implementation, which greatly facilitates the combined analysis. Ghosh et. al. [56] studied the stochastic sensitivity analysis of structures, based on the first-order perturbation method, they show that the stochastic structural response sensitivity is quite satisfactory compared to Monte Carlo results under small variation of input random parameters. Bhattacharyya and Chakraborty [57] employed the Neumann expansion accompanied with Monte Carlo simulation (NE-MCS) for response sensitivity analysis.
within the framework of SFEM. By using the load averaging method and Cholesky decomposition for digital simulation, randomness of design variables is modeled. It is shown that response sensitivity analysis of the NE-MCS algorithm is more efficient and accurate than that of the direct Monte Carlo simulation and first-order perturbation technique. Two principles of the response sensitivity analysis are illustrated in the following review.

2.2.1 Perturbation method for stochastic sensitivity

The equilibrium equation, Eq.(2.1) of linear elastostatic structure discretized through FEM can be expressed as

\[
[K\{h\}][U\{h\}] = \{F\{h\}\}
\]  

(2.20)

where \([K\{h\}]\) is the global stiffness matrix, \([U\{h\}]\) is the response vector and \([F\{h\}]\) is the force vector. They are functions of \([h]\) which is the vector of design variables of size \(m \times 1\). In deterministic sensitivity analysis, differentiating Eq.(2.20) with respect to ‘\(k\)-th’ design variable ‘\(h_k\)’ results

\[
\begin{align*}
[K]\left(\frac{\partial U}{\partial h_k}\right) + \left(\frac{\partial K}{\partial h_k}\right)[U] &= \left(\frac{\partial F}{\partial h_k}\right) \\
\left(\frac{\partial U}{\partial h_k}\right) &= [K]^{-1}\left[\left(\frac{\partial F}{\partial h_k}\right) - \left(\frac{\partial K}{\partial h_k}\right)[U]\right]
\end{align*}
\]

(2.21)

A simplified form can be written as

\[
\{y_k\} = [K]^{-1}\{F_k\}
\quad \text{or} \quad
[K]\{y_k\} = \{F_k^*\}
\]

(2.22)
where \( \{ y_k \} \) is the displacement sensitivity vector with respect to \( h_k \), \( \{ y_k \} = \frac{\partial}{\partial h_k} \{ U \} \), \( \{ F_k^* \} \)
is defined as pseudo-force vector

\[
\{ F_k^* \} = \frac{\partial}{\partial h_k} \{ F \} - \{ U \} \frac{\partial}{\partial h_k} \{ K \} \quad (2.23)
\]

To obtain the stochastic sensitivity of the structural response, \( \{ K \} \), \( \{ y_k \} \) and \( \{ F_k^* \} \) in Eq.(2.22) are expanded via Taylor series

\[
\{ F_k^* \} = \left( \sum_{i=1}^{N} \frac{\partial \{ F_k^* \}}{\partial h_k} \Delta h_k \right) + ...
\]

Substituting Eq.(2.24) into Eq.(2.20) and equating terms of same order and neglecting second and higher-order terms in expressions, following set of recursive equations can be obtained

Zero-order:

\[
\{ K \} \{ y_k \} = \{ F_k^* \} \quad (2.25)
\]

First-order:

\[
\left[ \frac{\partial \{ F_k^* \}}{\partial h_k} \right] = \frac{\partial \{ F_k^* \}}{\partial h_k} \quad (2.26)
\]

where Eq.(2.25) contributes to the mean part and Eq.(2.26) to the deviatoric part of stochastic response sensitivity.

In case of first-order perturbation, response sensitivity statistic can be obtained using expansion of \( \{ y_k \} \) in Eq.(2.24) with following expected value and covariance matrix
\[ E\left[ \{y_k\} \right] = \{\bar{y}_k\} \]

\[ \text{Cov}\left( \{y_k\}, \{y_k\}^T \right) = E\left[ \left( \{y_k\} - \{\bar{y}_k\} \right) \left( \{y_k\} - \{\bar{y}_k\} \right)^T \right] = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial \{y_k\}}{\partial h_i} \frac{\partial \{y_k\}^T}{\partial h_j} E[\Delta h_i \Delta h_j] \quad (2.27) \]

The stochastic sensitivity of the first-order perturbation technique can obtain acceptable accuracy for small fluctuation in random variables. The numerical algorithm is shown to be readily adaptable to existing finite element program since the models of stochastic finite element and stochastic design sensitivity are almost identical. However, for higher coefficient of variation, the perturbation is not so effective.

2.2.2 NE-MCS technique for stochastic sensitivity

To obtain sensitivity of response with respect to \( h_i \), substituting Eq.(2.14) into Eq.(2.22) yields

\[ \{y_k\} = [K]^{-1} \{F^*_k\} = (I - P + P^2 - P^3 + ...) [K]^{-1} \{F^*_k\} = \{y_{k0}\} + \{y_{k2}\} - \{y_{k3}\} + ... \quad (2.28) \]

where \( \{y_{k0}\} = [K]^{-1} \{F^*_k\} \) is the mean part and other terms of expansion contribute to the fluctuating component of the random sensitivity. The above series can be computed by solving the following recursive equation.

\[ [K] \{y_{k,i}\} = [\Delta K] \{y_{k,(i-1)}\} \quad (2.29) \]

The expansion series in Eq.(2.29) may be terminated after a few terms when the convergence and desired accuracy of the solution is obtained.

The features of stochastic sensitivity obtained from NE-MCS technique are similar with that of the NE-MCS stochastic analysis.
2.3 Stochastic optimization of structures

In the last decades, optimization under uncertainty has experienced rapid development in both theory and algorithm. Its approaches include expectation minimization, minimization of deviations from goals, minimization of maximum costs and optimization over soft constraints.

Introducing uncertainty within the deterministic optimization framework allows for the more realistic results, but also adds the complexity of optimization, makes the optimization of structures highly nonlinear. To solve the stochastic nonlinear optimization problem, there are two fundamental techniques. The first one is the regularized decomposition. Ruszczynski [58] added a quadratic terms to the objective function for improving convergence of the L-shaped decomposition method. Taking advantage of the augmented Lagrangian method, Dempster [59] added a quadratic penalty to ensure convexity, yielding more efficient computation. Rockafellar and Wets [60] also developed a similar method, the Progressive Hedging Algorithm. Their significant limitations are on either the objective function type or the underlying distributions for the uncertain variables.

Another technique is to identify problem of specific structures and transform the problem into a deterministic nonlinear programming problem. For example, Charnes and Cooper [61] replaced the uncertain constraints with the appropriate probabilities expressed in terms of moments. In Chance Constrained Programming, the uncertainty distributions should be stable; the uncertain variables are linear in the chance constraint. The problem needs to satisfy the general convexity conditions. Recently Marti [62] converts the random problems in plastic structural analysis and optimal design into
deterministic substitute problems by applying stochastic optimization methods. The advantage of these methods is that one can apply the deterministic optimization techniques to solve the stochastic optimization problem.

In the deterministic nonlinear optimization, the sequential quadratic programming (SQP) algorithms are widely considered today as the most effective general techniques for solving nonlinear programming with nonlinear constraints numerical experiments also show that SQP methods have good convergence properties. The idea of solving nonlinear problem by a sequence of quadratic programming subproblem was first suggested Wilson [63]. SQP methods were popularized mainly by Biggs [64], Han [65] and Powell [66,67], Not only is SQP very efficient for solving nonlinearly constrained optimization problems, but also has been applied to some problems such as nonsmooth equations, variational inequality problems, mathematical programs with equilibrium constraints, etc. and some large-scale problem.

Since the stochastic optimization problem is essentially a nonlinear program, the solution methods for nonlinear programming problems can be applied, and some recent studies indicated that sequential quadratic programming (SQP) is a very promising method particularly for solving nonlinear programs [68-72]. Moreover, the SQP method had been successfully adapted for solving optimal design of structures based on the perturbation method. Lee and Lim [73] presented a general formulation of the optimism design problem with the random parameter as followings

\[
\text{Minimize } F(x^0) \\
\text{Subject to the equilibrium equations}
\]

\[
K^0 Z^0 = f^0
\]
and

\[ (K^0 - \lambda^0 M^0)\phi^0 = 0 \]  

(2.31b)

and the normality condition:

\[ \phi^T M \phi = 1 \]  

(2.32)

the first-order perturbed equations

\[ K^0 Z_k^I = f_k^I - K_k^I Z^0 \]  

(2.33a)

and

\[ (K^0 - \lambda^0 M^0)\phi_k^I = \left(-\left(K_k^I - \lambda_k^I M^0 - \lambda^0 M_k^I\right)\phi^0 \right) \]  

(2.33b)

the second-order perturbed equations

\[ K^0 Z_{kl}^{II} = f_{kl}^{II} - K_k^I Z_k^I - K_{kl}^{II} Z^0 \]  

(2.34a)

and

\[ (K^0 - \lambda^0 M^0)\phi_{kl}^{II} = \left(-\left(K_k^I - \lambda_k^I M^0 - \lambda^0 M_k^I\right)\phi_k^I - \left(K_i^I - \lambda_i^I M^0 - \lambda^0 M_i^I\right)\phi_i^I \right) \]  


and

\[ G_i \leq 0 \quad i = 1, 2, \ldots \]  

(2.35)

and the bounds \( x^{0L} \leq x^0 \leq x^{0U} \)  

(2.36)

where \( F(x^0) \) is taken as expected cost function. \( x_0 \) are design variables. \( G_i \) represents limits on stresses, displacements, and frequencies of the structures. The formulation is based on the stochastic finite element method. It takes into full account the stress, displacement, and natural frequency constraints with the random variables. A gradient nonlinear programming technique, PLBA, based on the quadratic programming method, is used to solve the problem.
Sedaghati and Esmailzadch [74] developed a new structural analysis and optimization algorithm to determine the minimum-weight of structures with the truss and beam-type members under displacement and stress constraints, by using the force method. The algorithm is based on the sequential quadratic programming (SQP) technique, and the finite element technique is based on the integrated force method. The objective function is defined mathematically as minimizing the structural mass:

$$\text{Min } M(A) = \sum_{i=1}^{n} \rho_i L_i A_i$$  \hspace{1cm} (2.37)

Subject to the \((n + m)\) stress and the displacement constraints

$$g_i(A) = \left| \frac{\sigma_i}{\bar{\sigma}_i} \right| - 1 \leq 0 \quad i = 1, 2, \ldots n$$

$$g_j(A) = \left| \frac{u_j}{\bar{u}_j} \right| - 1 \leq 0 \quad i = 1, 2, \ldots m$$  \hspace{1cm} (2.38)

and the bounds

$$\bar{A}_i \leq A_i \quad i = 1, 2, \ldots n$$  \hspace{1cm} (2.39)

where \(M(A)\) is the total mass of the structure. \(\bar{A}_i\) is the lower limit on the \(i\)-th design variable, \(\bar{\sigma}_i\) and \(\bar{u}_j\) are the allowable limit value of the \(j\)-th stress and displacement, respectively. It is found that the optimization technique based on the IFM is computationally more efficient than the displacement method.

Chen [75] proposed a stochastic optimization modeling procedure to improve product quality by reducing variations. He developed two stochastic versions of SQP respectively embedded with a Monte Carlo simulation and numerical approximation in
dynamic-characteristic robust design. Fares, et. al. [76] provided a sequential semidefinite programming, an extension of SQP, to solve the robust control problems.

To improve the quality of a product caused by variation without eliminating these causes, the robust design is introduced to reduce the effects of variability. Doltsinis and Kang [77] presented a formulation for the robust design of structures with stochastic parameters. Both the expected value and the standard deviation of the objective function are to be minimized in the optimization. The formulations of robust design are given as following

1) minimize \( \{E(f(d)), \sigma(f(d))\} \) 
subject to \( E(g_i(d)) + \beta_i \sigma(g_i(d)) \leq 0 \quad (i = 1, 2, \ldots k) \)
\( \sigma(h_j(d)) \leq \sigma_j \quad (i = 1, 2, \ldots l) \)
\( d^- \leq d \leq d^+ \) \( (2.40) \) \( (2.41) \)

where \( d \) can be either deterministic or random design variables. \( \beta_i, \beta_i > 0 \) is a prescribed feasibility index for the \( i \)-th original constraint.

2) minimize \( \tilde{f} = (1 - \alpha) \frac{E(f(d))}{\mu^*} + \alpha \frac{\sigma(f(d))}{\sigma^*} \) 
subject to \( E(g_i(d)) + \beta_i \sigma(g_i(d)) \leq 0 \quad (i = 1, 2, \ldots k) \)
\( \sigma(h_j(d)) \leq \sigma_j \quad (i = 1, 2, \ldots l) \)
\( d^- \leq d \leq d^+ \) \( (2.42) \) \( (2.43) \)

Here, \( \alpha, 0 < \alpha < 1 \) is the factor weighting the two objectives, \( \mu^* \) and \( \sigma^* \) are normalization factors. The robust design optimization problem has been solved with SQP.
Lee and Park [78] provided a multiobjective function defined by the mean and the standard deviation of the original objective function. To obtain the optimum value insensitive to variations of design variable, the constraints are supplemented by adding a penalty term to the original constraints. Levi et. al. [79] introduced a multi-objective optimization method suitable for dealing with stochastic systems. The optimization method can manage the uncertain system parameters and external disturbances introduced by means of a Gaussian stochastic process.

Parkinson et. al. [80] proposed a general approach for robust optimal design and addressed the design feasibility and the control the transmitted variation and also discussed the feasibility robustness and sensitivity robustness for robust mechanical design. Sundaresan et. al. [81] applied a sensitivity index optimization approach to determine a robust optimum. Mulve et. al. [82] treated robust design as a bi-objective non-linear programming problem and employed a multi-criteria optimization approach to generate a complete and deficient solution set to support decision-making. Chen et. al. [83] developed a robust design methodology to minimize variations caused by the noise and control factors.

The various objective functions, analysis techniques and applications used for robust design are reviewed by Zang [84]. Mattson and Messac [85] also give a brief survey on how various robust design optimization methods handle constraint condition.
To the best of our knowledge, the existing stochastic finite element method is basically a combination of the deterministic finite element method, perturbation technique and stochastic analysis, that is, the SFEM incorporate various classes of uncertainties into structural mechanics in more realistic, natural and concise manner with the aid of the finite element method and Taylor series expansion methods. Analogous to the stochastic finite element formulation in the displacement method, the integrate force method, as an alternative to the stiffness method, is also extended to the stochastic analysis in the linear static domain.

Therefore, this chapter first gives a brief introduction of the (primal) integrated force method (IFM), the dual integrated force method (IFMD) and the variational energy formulation in the deterministic analysis. In section 3.2, the formulations of stochastic analysis for IFM/IFMD using the first- and second-order perturbation approach are presented. In section 3.3, a stochastic variational formulation of IFM is developed. In section 3.4, a stochastic formulation of IFM/IFMD using the Neumann expansion technique is provided. The last section devotes to the stochastic sensitivity analysis of IFM.
3.1 Deterministic Formula

Structural response can be calculated by using the primal integrated force method (IFM) as well as the dual integrated force method (IFMD). Both IFM and IFMD equations are described next, and a variational functional is also introduced.

3.1.1 Integrated force method

The governing equation of IFM for a continuum is discretized by a finite element model with n force and m displacement unknowns. The m equilibrium equations

\[
[B][F] = \{P\}
\]

(3.1)

and the \( r = (n - m) \) compatibility conditions

\[
[C][G][F] = \{\delta R\}
\]

(3.2)

are coupled to obtain the IFM governing equation in static analysis as

\[
\begin{bmatrix}
[B] \\
[C][G]
\end{bmatrix}[F] = \begin{bmatrix}
\{P\} \\
\{\delta R\}
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
S
\end{bmatrix}[F] = \{P^*\}
\]

(3.3)

where \([B]\) is the \((m \times n)\) equilibrium matrix, \([C]\) is the \((r \times n)\) compatibility matrix, \([G]\) is the \((n \times n)\) concatenated flexibility matrix, \(\{P\}\) is the m components load vector. Note that for simplicity, the star is dropped out \(\{P^*\}\) in later formulas. \(\{\delta R\}\) is the \(r\)-component effective initial deformation vector

\[
\{\delta R\} = -[C]\{\beta^0\}
\]

(3.4)

Here, \(\{\beta^0\}\) is the n-component initial deformation vector and \([S]\) is the \((n \times n)\) IFM governing matrix.
The solution of Eq.(3.1) yields the n forces, \( \{F\} \). The \( m \) displacements, \( \{X\} \), are obtained from the force \( \{F\} \) by backsubstitution

\[
\{X\} = [J][G]\{F\} + \{\beta^0\}
\]

where \( [J] \) is the \( (m \times n) \) deformation coefficient matrix defined as

\[
[J] = \text{m rows of } [S]^{-1}
\]

Equation (3.3) and (3.5) represent the two key IFM relations for finite element analysis in structural analysis.

It must be stressed that the stochastic characteristic of the matrices and vectors are described as follows:

1. \([B]\), \([C]\) and \([J]\) are the deterministic matrices, because their components only include the non-stochastic geometrical parameters.

2. \([G]\), \([S]\), \(\{P\}\) and \(\{R\}\) are the stochastic matrices and vectors, respectively. They are derived from the stochastic parameters of material properties, sizing design variables and uncertain loads, respectively.

3.1.2 Dual integrated force method

The dual integrated force method is obtained from the IFM equation by mapping forces into displacements at the element level. Analogous to the IFM, the displacement may be calculated by a symmetrical set of equations.

\[
[D]\{X\} = \{P^D\}
\]
where $[D]$ is the $(m \times m)$ symmetrical dual matrix and $[D] = [B]G^{-1}[B]^T$. It is a stochastic matrix. $\{P^D\}$ is the $m$ component dual load vector and $\{P^D\} = \{P\} + [B]G^{-1}\{\beta^0\}$. It is a stochastic vector.

From the displacement, the force is back-calculated as

$$\{F\} = [G]^{-1}([B]^T\{X\} - \{\beta^0\})$$

Eq. (3.7) closely resembles the equations of the popular and stiffness method. IFM and IFMD are analytically equivalent and produce identical responses. For design and sensitivity analysis, the primal IFM, however, has some advantages over the IFMD [8, 11].

3.1.3 Stress and strain formulas

It is seen from Eq. (3.1) and (3.2) that the equation system simultaneous satisfies the equilibrium equations and the compatibility conditions is first established on the element level, and then the assembly procedure is used to derive the system Eq. (3.3).

In the IFM [86,87], the independent displacement and stress interpolations are used to obtain the following expressions:

$$\{u\} = [N][U_e]$$

$$\{\sigma\} = [Y][F_e]$$

where $\{u\}$, $\{u\} = \{u_x, u_y, u_z\}^T$, is the displacement vector. $\{U_e\}$ is the vector of displacements at element nodes resulting from the finite element discretization. $\{\sigma\}$, $\{\sigma\} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}^T$, is the vector of stress components. $\{F_e\}$ is the vector of
element independent generalized force. \([N]\) is the matrix of displacement interpolation functions. \([Y]\) is the stress interpolation matrix.

The strain vector, \(\{\varepsilon\}\), \(\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\}\), is obtained by differentiation of the displacement field and is given by

\[
\{\varepsilon\} = [Z][U_e]
\]

(3.11)

where \([Z]\) is the matrix of differential operator.

3.1.4 Variational energy formulation of IFM

The variational energy principle [88] of the IFM can be expressed by

\[
\pi_s = A + B - W
\]

(3.12)

where \(A(\sigma, u)\) represents the strain energy, \(B(\varepsilon, \sigma^e)\) is the complementary strain energy functional and \(W(P, u)\) is the potential of the external forces.

For three-dimensional elasticity problem, the functional \(A(\sigma, u)\), \(B(\varepsilon, \sigma^e)\) and \(W(P, u)\) can be written as

\[
A(\sigma, u) = \int_{\Omega} \left[ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \tau_{yz} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \tau_{zx} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] d\Omega
\]

(3.13a)

\[
B(\varepsilon, \sigma^e) = \int_{\Omega} \left[ \varepsilon_x \sigma_x^e + \varepsilon_y \sigma_y^e + \varepsilon_z \sigma_z^e + \gamma_{xy} \tau_{xy}^e + \gamma_{yz} \tau_{yz}^e + \gamma_{zx} \tau_{zx}^e \right] d\Omega
\]

(3.13b)

\[
W(P, u) = \int_{\partial \Omega} \left[ (P_x \bar{u} + P_y \bar{v} + P_z \bar{w}) d(\partial \Omega) \right] + \int_{\partial \Omega} \left[ (\overline{P_x} u + \overline{P_y} v + \overline{P_z} w) d(\partial \Omega) \right] + \int_{\Omega} \left[ (B_x u + B_y v + B_z w) \right] d\Omega
\]

(3.13c)
where $\sigma_x^e, \sigma_y^e, \sigma_z^e, \tau_{xy}^e, \tau_{yz}^e,$ and $\tau_{zx}^e$ are the six redundant stress components. $B_x, B_y$ and $B_z$ are three body force components. $P_x, P_y$ and $P_z$ are three traction components. $\overline{P_x}, \overline{P_y}$ and $\overline{P_z}$ are the three specified external loads on the boundary. $\overline{u}, \overline{v}$ and $\overline{w}$ are three prescribed displacements on the boundary. $u, v$ and $w$ are three displacement components.

The variational formulation of IFM has three properties as following:

1. The value of the variational formulation at the stationary point is zero.

$$\left. \pi_S \right|_{SP} = 0 \quad (3.14)$$

2. The potential energy functional $\pi_p$ can be generated from the variational formulation for IFM. If forces in $\pi_S$ are eliminated in favor of displacements

$$\left. \pi_S \right|_{F \rightarrow X} = \pi_p \quad (3.15)$$

3. The complementary energy functional $\pi_C$ of the standard force method can be generated as a special case of $\pi_S$ by introducing the concept of redundant forces explicitly as

$$\{F\} = \{F_0^0\} + [C]^T \{R\} \quad \{F_0^0\} = [B_o^{-1}] \{P\} \quad [B] = [B_o : B_c]$$

then

$$\left. \pi_S \right|_{F \rightarrow R \rightarrow X} = \pi_C \quad (3.16)$$

3.2 Stochastic analysis for structure response in IFM/IFMD

Taking into consideration that parameters of structures may be described in probabilistic terms, the random response of the stochastic structural system will be determined by using IFM/IFMD. The most widely used procedure for evaluating the
stochastic response is the well established perturbation approach. In the perturbation approach, the random functions are expressed as the sum of deterministic and random components. The first- and second-order perturbation approaches embedded into IFM/IFMD are developed to the stochastic analysis for structural responses as following.

3.2.1 System Description

In measuring experiments, the mean value and the covariance value of the structural parameters can be statistically obtained. These parameters are the basic ones for the stochastic structural analysis. Here they are divided into four categories,

1. Load vector, \( \{P\} \), consists of mechanical load, thermal load, and setting of support load. It is defined in terms of a mean vector \( \{\mu^P\} \) and a covariance matrix \( \text{Cov}(\{P\}) \).

2. Material properties include elastic modulus, \( E \), Poisson’s ratio, \( \nu \), coefficient of thermal expansion, \( \alpha \), and material density, \( \rho \). The corresponding statistical parameters are the mean value \( \{\mu^m\} \) and the associated covariance matrix.

3. Sizing design parameters include area of bar, moment of inertia of beam and thickness of plate. They are defined through the corresponding statistical values.

4. Geometrical parameters such as the length of a bar or beam and span of a plate are considered as deterministic variables here. They are given by the nominal value.

To make use of an additive decomposition of any physical property of interest into a deterministic and a random part, the structural parameters in stochastic analysis are assumed to very spatially as a two-dimensional homogeneous stochastic field. For
example, the coefficient of thermal expansion, $\alpha$, with a mean value $\mu_\alpha$ and a standard deviation $\sigma_\alpha$, may be written as

$$\alpha(x) = \mu_\alpha \left(1 + q_\alpha(x) \right) \quad \text{or} \quad q_\alpha(x) = \frac{\alpha - \mu_\alpha}{\mu_\alpha}$$  \hspace{1cm} (3.17)$$

where $q_\alpha(x)$ is the fluctuating component of the coefficient of thermal expansion, also called the normalized primitive random variable. $x = [x, y]^T$ indicates the position vector.

Therefore, the mean value of $q_\alpha(x)$ is obtained by

$$\mu_{q_\alpha(x)} = E[q_\alpha(x)] = E\left[ \frac{\alpha - \mu_\alpha}{\mu_\alpha} \right] = \frac{E[\alpha] - \mu_\alpha}{\mu_\alpha} = \frac{\mu_\alpha - \mu_\alpha}{\mu_\alpha} = 0 \hspace{1cm} (3.18)$$

and the variance of $q_\alpha(x)$ is also given by

$$\sigma^2_{q_\alpha(x)} = E\left[ (q_\alpha(x) - \mu_{q_\alpha})^2 \right]$$

$$= E\left[ \left( \frac{\alpha - \mu_\alpha}{\mu_\alpha} \right)^2 \right]$$

$$= \frac{1}{\mu_\alpha^2} E\left[ (\alpha - \mu_\alpha)^2 \right]$$

$$= \frac{\sigma^2_\alpha}{\mu_\alpha^2} \hspace{1cm} (3.19)$$

and the covariance, $\gamma^{ij}_{q_\alpha(x)}$ of $q_{\alpha_i}(x)$ and $q_{\alpha_j}(x)$ is

$$\gamma^{ij}_{q_\alpha(x)} = E\left[ (q_{\alpha_i}(x) - \mu_{q_{\alpha_i}}) (q_{\alpha_j}(x) - \mu_{q_{\alpha_j}}) \right]$$

$$= E\left[ \left( \frac{\alpha_i - \mu_{\alpha_i}}{\mu_{\alpha_i}} \right) \left( \frac{\alpha_j - \mu_{\alpha_j}}{\mu_{\alpha_j}} \right) \right]$$

$$= \frac{1}{\mu_{\alpha_i} \mu_{\alpha_j}} E\left[ (\alpha_i - \mu_{\alpha_i}) (\alpha_j - \mu_{\alpha_j}) \right] \hspace{1cm} (3.20)$$

$$= \frac{\gamma^{ij}}{\mu_{\alpha_i} \mu_{\alpha_j}}$$
where the variance $\sigma_\alpha^2$ or covariance $\gamma_{\alpha}^{ij}$ of $\alpha$ is assumed to be much smaller than the square of its mean or the multiplying of two relative means. Hence, the second-moment characteristics of the coefficient of thermal expansion can be defined from the knowledge of the mean and covariance function of $q_\alpha(x)$. In a similar way, other random structural parameters are also defined by the normalized primitive random variables.

3.2.2 Perturbation technique and stochastic response using IFM/IFMD

Since the IFM method is applicable to the finite element discrete analysis in a well-conditioned system, the similar computational implementation in the stochastic finite element method (SFEM) can be directly used in the stochastic analysis of IFM/IFMD.

With the assumption of the small covariance of the random variables, a Taylor’s expansion, involving terms up to the second order with respect to the primitive random variables is employed for the calculation of stochastic response in the (primal) IFM and dual IFM. For example, the Taylor series expansion of the internal force vector, $\{F\}$, has the following form:

$$\{F\} = \overline{\{F\}} + \sum_{i=1}^{N} \{F_i\} q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_{ij}\} q_i q_j + ...$$  \hspace{1cm} (3.21)

where $\overline{\{F\}}$ is the force vector evaluated at $q_i = 0$ (i =1,2,…,N), $\{F_i\}$ and $\{F_{ij}\}$ are partial derivatives of $\{F\}$ defined as follows:

$$\{F_i\} = \left. \frac{\partial \{F\}}{\partial q_i} \right|_{\{q\} = \{0\}} \hspace{1cm} (3.22a)$$
\[ \{ F_{ij} \} = \frac{\partial^2 \{ F \}}{\partial q_i \partial q_j} \bigg|_{\{ q \} = \{ 0 \}} \]  

(3.22b)

Similarly, the expansions for \( \{ S \} \) and \( \{ P \} \) in the primal IFM governing equation are also obtained by

\[
\{ S \} = \left[ S \right] + \sum_{i=1}^{N} \left[ S_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ S_{ij} \right] q_{i} q_{j} + \ldots 
\]

\[
\{ P \} = \left[ P \right] + \sum_{i=1}^{N} \left[ P_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ P_{ij} \right] q_{i} q_{j} + \ldots 
\]

(3.23)

Substituting Eqs.(3.21) and (3.23) into Eq.(3.1) and retaining terms up to the second-order field,

\[
\left( \left[ S \right] + \sum_{i=1}^{N} \left[ S_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ S_{ij} \right] q_{i} q_{j} + \ldots \right) \left( \left[ F \right] + \sum_{i=1}^{N} \left[ F_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ F_{ij} \right] q_{i} q_{j} + \ldots \right) 
\]

\[= \left[ P \right] + \sum_{i=1}^{N} \left[ P_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ P_{ij} \right] q_{i} q_{j} + \ldots 
\]

\[
\left[ S \right] \left[ F \right] + \sum_{i=1}^{N} \left[ S_{j} \right] \left[ F_{i} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ S_{ij} \right] \left[ F_{i} \right] q_{i} q_{j} + \sum_{i=1}^{N} \left[ S_{j} \right] \left[ F \right] q_{i} + \sum_{i=1}^{N} \left[ S_{j} \right] \sum_{i=1}^{N} \left[ F_{ij} \right] q_{j} 
\]

\[+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ S_{ij} \right] q_{i} q_{j} \left[ F \right] + \ldots 
\]

\[= \left[ P \right] + \sum_{i=1}^{N} \left[ P_{j} \right] q_{i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ P_{ij} \right] q_{i} q_{j} + \ldots 
\]

(3.24)

By equating terms of equal orders, the zeroth-, first- and second-order equations corresponding to Eq.(3.3) can be written as follows:

zeroth order:

\[
\left[ S \right] \left[ F \right] = \left[ P \right]
\]

(3.25)

first order:

\[
\left[ S \right] \left[ F_{j} \right] + \left[ S_{j} \right] \left[ F \right] = \left[ P_{j} \right]
\]

(3.26)

second order:

\[
\left[ S \right] \left[ F_{ij} \right] + 2 \left[ S_{j} \right] \left[ F_{i} \right] + \left[ S_{ij} \right] \left[ F \right] = \left[ P_{ij} \right]
\]

(3.27)

Eqs. (3.25) – (3.27) are also evaluated by the following set of recursive equations:
\[
\overline{\{F\}} = S^{-1} \{p\}
\]
\[
\{F_i\} = S^{-1} \{(P_i) - [S_{ji}] \overline{\{F\}}\}
\]
\[
\{F_j\} = S^{-1} \{(P_j) - 2[S_{ij}] \{F_i\} - [S_{ij}] \overline{\{F\}}\}
\]

(3.28)

Therefore, the first-order approximation for the force can be obtained by truncating the right-hand side of Eq.(3.22) after the second term as

\[
\{F\} = \overline{\{F\}} + \sum_{i=1}^{N} \{F_i\} q_i
\]

(3.29)

Taking the expectation of Eq.(3.28) yields

\[
\mu'F = E' [\{F\}] = E \left[ \overline{\{F\}} + \sum_{i=1}^{N} \{F_i\} q_i \right] = \overline{\{F\}}
\]

(3.30)

and the covariance matrix of the force vector can be given by

\[
\text{Cov}' \left( \{F\}, \{F\}' \right) = E \left( \{(F) - E' [\{F\}](F) - E' [(F)]' \right)
\]

\[
= E \left[ (F) (F)' \right] - (F) (F)'
\]

\[
= E \left[ \left( \overline{\{F\}} + \sum_{i=1}^{N} \{F_i\} q_i \right) \left( \overline{\{F\}} + \sum_{i=1}^{N} \{F_i\} q_j \right)' \right] - (F) (F)'
\]

\[
= E \left[ \overline{\{F\}} (F)' + \sum_{i=1}^{N} \{F_i\} q_i (\overline{\{F\}})' + \sum_{i=1}^{N} \{F_i\} q_j (\sum_{i=1}^{N} \{F_i\} q_i)' - \overline{\{F\}} (F)' \right]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_i\} \{F_j\}' E[q_i q_j]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_i\} \{F_j\}' \text{Cov}(q_i, q_j)
\]

(3.31)

where \(E[q_i q_j] = \text{Cov}(q_i, q_j)\) due to \(E[q_i] = 0\) and \(E[q_j] = 0\). \(\text{Cov}(q_i, q_j)\) is the element of the covariance matrix for \(\{q\}\).

Then, the second-order approximation for the force vector can be written as
\begin{equation}
\{F\} = \{\bar{F}\} + \sum_{i=1}^{N} \{F, q_i\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\}
\end{equation}

(3.32)

with the expected value

\begin{equation}
\mu^q = E^q[\{F\}] = E\left[\{\bar{F}\} + \sum_{i=1}^{N} \{F, q_i\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\}\right] = \{\bar{F}\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, \mu E[q_i q_j]\}
\end{equation}

(3.33)

and covariance matrix

\begin{equation}
\text{Cov}^q (\{F\}, \{F\}^T) = E[\{F\} \{F\}^T] - E[\{F\}] E[\{F\}]^T
\end{equation}

\begin{align}
= & E \left[ \{\bar{F}\} + \sum_{i=1}^{N} \{F, q_i\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \right] \left[ \{\bar{F}\} + \sum_{i=1}^{N} \{F, q_i\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \right]^T \\
- & \left( \{\bar{F}\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i, q_j\} \right) \left( \{\bar{F}\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i, q_j\} \right)^T \\
= & E \left[ \{\bar{F}\} \{\bar{F}\}^T + \sum_{i=1}^{N} \{F, q_i\} \sum_{i=1}^{N} \{F, q_i\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \sum_{j=1}^{N} \{F, q_i q_j\}^T \right] \\
+ & \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \sum_{i=1}^{N} \{F, q_i\} \sum_{i=1}^{N} \{F, q_j\} \left( \sum_{i=1}^{N} \{F, q_i\} \sum_{i=1}^{N} \{F, q_j\} \right)^T \\
- & \left[ \{\bar{F}\} \{\bar{F}\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i, q_j\} \sum_{j=1}^{N} \{F, q_i, q_j\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \sum_{j=1}^{N} \{F, q_i q_j\}^T \right] \\
+ & \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i, q_j\} \sum_{j=1}^{N} \{F, q_i, q_j\} \sum_{j=1}^{N} \{F, q_i, q_j\}^T \sum_{j=1}^{N} \{F, q_i, q_j\}^T \\
= & \{\bar{F}\} \{\bar{F}\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i, q_j\} \sum_{j=1}^{N} \{F, q_i, q_j\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \sum_{j=1}^{N} \{F, q_i q_j\}^T \\
+ & \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \{F, q_i, q_j, q_k\} \sum_{j=1}^{N} \{F, q_i, q_j, q_k\}^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i q_j\} \sum_{j=1}^{N} \{F, q_i q_j\}^T \\
+ & \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \{F, q_i, q_j, q_k, q_l\} \sum_{j=1}^{N} \{F, q_i, q_j, q_k, q_l\}^T \\
= & \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i\} \sum_{j=1}^{N} \{F, q_j\} E[q_i q_j] \\
= & \sum_{i=1}^{N} \sum_{j=1}^{N} \{F, q_i\} \sum_{j=1}^{N} \{F, q_j\} \text{Cov}(q_i, q_j)
\end{align}

(3.34)
Assuming $E[q_i q_j q_k] = 0$ and $\text{Cov}(q_i, q_j, q_k, q_l) = 0$, i.e. $E[q_i q_j q_k q_l] = E[q_i] E[q_j] E[q_k] E[q_l]$.

In a manner similar to that for the force, the expected values and covariance matrices of the displacement, stress and strain in IFM/IFMD are also obtained analytically by using Eqs. (3.5), (3.7), (3.8), (3.9) and (3.11). These stochastic analytical formulations are shown in Table 3.1. Note that the zeroth-order equations are equal to the deterministic governing equations, the first- and second-order equations consist of some recursive items in the equations. It’s easy to find that the first- or second-order approximation expressions of the responses in the IFM/IFMD are similar. Moreover, the covariance matrices of the first-order approximation are equal to that of the second-order approximation.

It is observed that the stochastic analysis formulations of response in IFM/IFMD are identical to the deterministic counterparts, which are the combination of deterministic and random components by using the perturbation approach. This analytical method follows all the steps of a conventional deterministic analysis and efficiently uses the existing techniques and algorithms. The variance of the uncertainties, however, must be small enough to obtain the acceptable accuracy using only a finite terms in the above series.

Because of the complexity of the stochastic analysis, the general computation coding for IFM analyzer requires extensive effect. The formulas for the stochastic analysis for IFM/IFMD had been programmed in closed form using the Macsyma, Maple software and FORTRAN codes. This yields the man values, covariance matrices and sensitivity analysis for response in the structure. A lot of structures have been also tested in Chapter 5.
Table 3.1 Stochastic analysis formulation for IFM/IFMD

<table>
<thead>
<tr>
<th>Methods</th>
<th>IFM</th>
<th>IFMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sys. Eqs (1)</td>
<td>$[S]F = [P]$</td>
<td>$[D]X = [P D]$</td>
</tr>
<tr>
<td>$\varepsilon^0$</td>
<td>$[\tilde{F}] = [\tilde{S}]^{-1}[\tilde{P}]$</td>
<td>$[\tilde{X}] = [\tilde{D}]^{-1}[\tilde{P} D]$</td>
</tr>
<tr>
<td>$\varepsilon^1$</td>
<td>$[F_j] = [S]^{-1}((P_j) - [S]f_j)$</td>
<td>$[X_j] = [D]^{-1}((P_{j0}) - [D]X)$</td>
</tr>
<tr>
<td>$\varepsilon^2$</td>
<td>$[F_{ij}] = [S]^{-1}((P_{ij}) - 2[S]f_{ij} - [S]f_j)$</td>
<td>$[X_{ij}] = [D]^{-1}((P_{ij0}) - 2[D]X_{ij} - [D]X)$</td>
</tr>
<tr>
<td>Sys. Eqs (2)</td>
<td>$[X] = [J][[G][F] + [P^0]]$</td>
<td>$[F] = [G]^{-1}([B^T][X] - [P^0])$</td>
</tr>
<tr>
<td>$\varepsilon^0$</td>
<td>$[\tilde{X}] = [J][[G][\tilde{F}] + [\tilde{P}^0]]$</td>
<td>$[\tilde{F}] = [G]^{-1}([B^T][\tilde{X}] - [\tilde{P}^0])$</td>
</tr>
<tr>
<td>$\varepsilon^1$</td>
<td>$[X_j] = [J][[G][F_j] + [G_j][F]] + [P_{j0}]$</td>
<td>$[F_j] = [G]^{-1}([B^T][X_j] - [P_{j0}] - [G_j][F])$</td>
</tr>
<tr>
<td>$\varepsilon^2$</td>
<td>$[X_{ij}] = [J][[G][F_{ij}] + [G_j][F_j] + [G_{ij}][F_j] + [P_{ij0}]$</td>
<td>$[F_{ij}] = [G]^{-1}([B^T][X_{ij}] - [P_{ij0}] - 2[G_j][F_j] - [G_{ij}][F])$</td>
</tr>
</tbody>
</table>

1st Approx. | $\mu_f = E[F]$, $\mu_x = E[X]$, $\operatorname{Cov}(F, X^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [F_i][F_j]^T E[g_i q_j]$ | $\mu_f = E[F]$, $\mu_x = E[X]$, $\operatorname{Cov}(F, X^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [X_i][X_j]^T E[g_i q_j]$ |

2nd Approx. | $\mu_f = E''[F]$, $\mu_x = E''[X]$, $\operatorname{Cov}(F, X^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [F_i][F_j]^T E[g_i q_j]$ | $\mu_f = E''[F]$, $\mu_x = E''[X]$, $\operatorname{Cov}(F, X^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [X_i][X_j]^T E[g_i q_j]$ |

Sys. Eqs (3,4) | $[\sigma] = [Y][F]$ | $[\sigma] = [Z][X]$ |
| $\varepsilon^0$ | $[\tilde{\sigma}] = [\tilde{Y}][\tilde{F}]$ | $[\tilde{\sigma}] = [\tilde{Z}][\tilde{X}]$ |
| $\varepsilon^1$ | $[\sigma_j] = [Y][F_j] + [Y_j][F]$ | $[\sigma_j] = [Z][X_j] + [Z_j][X]$ |
| $\varepsilon^2$ | $[\sigma_{ij}] = [Y][F_{ij}] + 2[Y_j][F_j] + [Y_{ij}][F]$ | $[\sigma_{ij}] = [Z][X_{ij}] + 2[Z_j][X_j] + [Z_{ij}][X]$ |

1st Approx. | $\mu_{\sigma} = E'[\sigma]$, $\mu_{\sigma} = E'[\sigma]$, $\operatorname{Cov}(\sigma, \sigma^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [\sigma_i][\sigma_j]^T E[g_i q_j]$ | $\mu_{\sigma} = E'[\sigma]$, $\mu_{\sigma} = E'[\sigma]$, $\operatorname{Cov}(\sigma, \sigma^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [\sigma_i][\sigma_j]^T E[g_i q_j]$ |

2nd Approx. | $\mu_{\sigma} = E''[\sigma]$, $\mu_{\sigma} = E''[\sigma]$, $\operatorname{Cov}(\sigma, \sigma^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [\sigma_i][\sigma_j]^T E[g_i q_j]$ | $\mu_{\sigma} = E''[\sigma]$, $\mu_{\sigma} = E''[\sigma]$, $\operatorname{Cov}(\sigma, \sigma^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} [\sigma_i][\sigma_j]^T E[g_i q_j]$ |
3.2.3 Simplification on computation of stochastic analysis

It may be noted from Eqs.(3.28) to (3.34) that the number of vector and matrix multiplications are proportional to \( N(N+1)/2 \), since the random variable matrix is usually symmetrical and the computations are compatible with the elemental discretization and nodal assembly procedures. This would be unacceptably expensive owing to the appearance of the double summations with \( i \) and \( j \) in Eqs.(3.31) to (3.34).

In order to solve this problem, an approach based on the diagonalization [41] may be used. It can transform the base of the system with correlated random variables into an equivalent system in the corresponding standard normal space with uncorrelated random variables only. Thus, through the standard eigenproblem, the covariance matrix \( \text{Cov}(q_i, q_j) \) is transformed to a diagonal variance matrix \( \text{Var}(c_i, c_j) \) as

\[
\psi \text{Cov}(q_i, q_j) = \text{Var}(c_i, c_j) \psi \quad i, j = 1, 2, \ldots, N
\]

where the covariance matrix, \( \text{Cov}(q_i, q_j) \), is assumed to be positive definite. \( \text{Var}(c_i, c_j) \) is an \( N \)-dimensional diagonal matrix defined as,

\[
\text{Var}(c_i, c_j) = \begin{cases} 
0 & \text{for } i \neq j \\
\text{Var}(c_i) & \text{for } i = j
\end{cases}
\]

and \( \psi \) is an \( N \times N \) orthonormal fundamental matrix, i.e.,

\[
\psi \psi^T = \psi^T \psi = I
\]

and there exists the following relationships,

\[
\text{Cov}(q_i, q_j) = \psi^T \text{Var}(c_i, c_j) \psi
\]

\[
c = \psi^T q \quad q = \psi^T c
\]
where $I$ is the $N \times N$ identity matrix and $c$ is a transferred $N \times 1$ random variable vector. The mean and variance of $c$ can be expressed in terms of $q$ by using the superposition technique as

$$
c_i = \sum_{j=1}^{N} \psi_{ji} q_j
$$

$$
E[c] = \psi^T E[q]
$$

$$
Var(c) = Var(c_i)
$$

(3.39)

Meanwhile, the first and second derivatives of an function $\cdot$ with respect to $c_i$ are obtained in terms of $q_i$ as

$$
\frac{\partial \cdot}{\partial c_i} = \psi_{ji} \frac{\partial \cdot}{\partial q_j}
$$

$$
\frac{\partial^2 \cdot}{\partial c_i^2} = \psi_{ji} \frac{\partial^2 \cdot}{\partial q_j \partial q_k} \psi_{ki} \quad i, j, k = 1, 2, \ldots, N
$$

(3.40)

where

$$
\psi_{ji} = [\psi_{j1}, \psi_{j2}, \ldots, \psi_{jN}]
$$

$$
\frac{\partial^2 \cdot}{\partial q_j \partial q_k} = \begin{bmatrix}
\frac{\partial^2 \cdot}{\partial q_1^2} & \frac{\partial^2 \cdot}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 \cdot}{\partial q_1 \partial q_N} \\
\frac{\partial^2 \cdot}{\partial q_2 \partial q_1} & \frac{\partial^2 \cdot}{\partial q_2^2} & \cdots & \frac{\partial^2 \cdot}{\partial q_2 \partial q_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \cdot}{\partial q_N \partial q_1} & \frac{\partial^2 \cdot}{\partial q_N \partial q_2} & \cdots & \frac{\partial^2 \cdot}{\partial q_N^2}
\end{bmatrix}
$$

(3.41)

To express the major characteristics of the probabilistic distributions of the discretized random field $q$ with an acceptable accuracy, only a few modes are required and the highest eigenvalues are dominant.
By substituting Eqs.(3.40) and (3.41) into Eqs.(3.25 to 3.27), we arrive at the following equations for the equivalent uncorrelated system:

Zeroth-order ($\varepsilon^0$):

$$\{\overline{F}(c_i)\} = \{\overline{S}(c_i)\}^{-1}\{\overline{P}(c_i)\}$$

First-order ($\varepsilon^1$):

$$\{F_j(c_i)\} = \{\overline{S}(c_i)\}^{-1}(\{P_j(c_i)\} - \{S_j(c_i)\}\{\overline{F}(c_i)\})$$

Second-order ($\varepsilon^2$):

$$\{F_{jj}(c_i)\} = \{\overline{S}(c_i)\}^{-1}(\{P_{jj}(c_i)\} - 2\{S_j(c_i)\}\{F_i(c_i)\} - \{S_{jj}(c_i)\}\{\overline{F}(c_i)\})$$

Therefore, the first-order approximation for the force can be written in terms of the uncorrelated random variables $c_i$ as

$$E'[\{F(c_i)\}] = \{\overline{F}(c_i)\}$$

and the second-order approximation for the force vector

$$E''[\{F(c_i)\}] = \{\overline{F}(c_i)\} + \frac{1}{2} \sum_{j=1}^{N} \{F_{jj}(c_i)\}\text{Var}(c_i)$$

Similarly, the first and second-order approximation for the displacement, stress and strain in IFM, can be given by

$$E'[\{X(c_i)\}] = \{\overline{X}(c_i)\}$$

$$E''[\{X(c_i)\}] = \{\overline{X}(c_i)\} + \sum_{j=1}^{N} \{X_{jj}(c_i)\}\text{Var}(c_i)$$
\[ E''[X(c_i)] = \mathbb{E}(X(c_i)) + \frac{1}{2} \sum_{i=1}^{N} \{X_{,i}\} \text{Var}(c_i) \]

\[ \text{Cov}''\left([X(c_i), X(c_i)^T]\right) = \sum_{i=1}^{N} \{X_{,i}(c_i)\} [X_{,i}(c_i)^T \text{Var}(c_i) \right) \] (3.47)

\[ E'[\sigma(c_i)] = \mathbb{E}(\sigma(c_i)) \]

\[ \text{Cov}'\left([\sigma(c_i), \sigma(c_i)^T]\right) = \sum_{i=1}^{N} \{\sigma_{,i}(c_i)\} [\sigma_{,i}(c_i)^T \text{Var}(c_i) \right) \] (3.48)

\[ E''[\varepsilon(c_i)] = \mathbb{E}(\varepsilon(c_i)) \]

\[ \text{Cov}''\left([\varepsilon(c_i), \varepsilon(c_i)^T]\right) = \sum_{i=1}^{N} \{\varepsilon_{,i}(c_i)\} [\varepsilon_{,i}(c_i)^T \text{Var}(c_i) \right) \] (3.49)

For IFMD, the similar formulas of displacement, force, stress, and strain can be also obtained. Here, we omit these formulas.

3.2.4 Neumann expansion technique for stochastic analysis using IFM/IFMD

In this section, an attempt is made to explore the Neumann expansion technique in deriving the IFM solution for the response variability within the framework of the Monte
Carlo method. It is important to note that the compatible combination of the Neumann expansion with Monte Carlo simulation works efficiently for computation of stochastic structural response \cite{6, 57} and avoids the repeated inversion of the random governing matrix $[S]$.

In the direct Monte Carlo simulation, the inversion of the random governing matrix in the IFM in each simulation needs to consume an enormous amount of CPU time. To efficiently solve the problem, the Choledky decomposition of $[S]$ is directly employed by, while the number of degree-of-freedom is large,

$$\begin{bmatrix} \mathbf{L} \\ \mathbf{L} \end{bmatrix}^T = [S]$$ \hspace{5cm} (3.50)

where $[L]$ is the lower triangular matrix. The unknown vector, $\{X\}$, and the internal force, $\{F\}$, in Eq.(3.3) can be easily obtained by

$$[L]\{X\} = \{P\} \hspace{5cm} (3.51a)$$

$$[L]\{F\} = \{X\} \hspace{5cm} (3.51b)$$

In the Neumann expansion technique, the governing matrix $[S]$ containing the spatial variabilities, can readily be decomposed into tow parts

$$[S] = \overline{[S]} + [\Delta S]$$ \hspace{5cm} (3.52)

where $\overline{[S]}$ is the governing matrix at the mean values. $[\Delta S]$ is the deviatoric part of $[S]$, $[\Delta S] = [S] - \overline{[S]}$.

Therefore, the random governing matrix is inverted by using the Neumann expansion,

$$[S]^{-1} = \left( \overline{[S]} + [\Delta S] \right)^{-1} = \left( [I] + [A] \right)^{-1} \overline{[S]}^{-1} = \left( [I] - [A] + [A]^2 - [A]^3 + \ldots \right) \overline{[S]}^{-1}$$ \hspace{5cm} (3.53)
where \([A] = \left[\overline{S}\right]^{-1}[\Delta S]\).

Substituting Eq.(3.53) into Eq.(3.5) the solution of the internal force, \(\{F\}\), is written by the series as

\[
\{F\} = \left(\left[I\right] + [A] \right)^{-1} \left[\overline{S}\right]^{-1} \{P\} \\
= \left(\left[I\right] - [A] + [A]^2 - [A]^3 + \ldots \right) \left[\overline{S}\right]^{-1} \{P\} \\
= \left(\left[I\right] - [A] + [A]^2 - [A]^3 + \ldots \right) \{F_0\} \\
= \{F_0\} - [A]\{F_0\} + [A]^2\{F_0\} - [A]^3\{F_0\} + \ldots \\
= \{F_0\} - \{F_1\} + \{F_2\} - \{F_3\} + \ldots 
\]

where \(\{F_0\} = \left[\overline{S}\right]^{-1}\{P\}\) is the deterministic mean part while \(\{F_1\}, \{F_2\}, \{F_3\}\) etc, are the stochastic part of the random force. The series solution can be expressed by the recursive equation as

\[
\left[\overline{S}\right]\{F_i\} = [\Delta S]\{F_{i-1}\} \quad i = 1, 2, \ldots 
\]

Thus, once \(\{F_0\}\) is obtained using the algorithm in Eq.(3.51) for \(\left[\overline{S}\right]\), then the random internal force for each simulation can be iteratively calculated with the aid of Eq. (3.55).

The expansion series in Eq.(3.54) may be terminated after a few terms if convergence of the series is conformed by using the criterion

\[
\frac{\left\|\sum_{k=0}^{i} (-1)^k \{F_k\}\right\|_2}{\|\{F_i\}\|_2} \leq \delta_{err} 
\]

where \(\delta_{err}\) is the allowable error to be specified for convergence (here \(\delta_{err} = 0.01\)). \(\|\|\) is the vector norm, \(\|\{F\}\|_2 = \sqrt{\{F\}^T \{F\}}\).
Other stochastic response like displacement, stress, and strain can be obtained by using Eq. (3.5), (3.10) and (3.11) and the obtained results $\{F\}$ like the direct MCS.

In similar manner, the stochastic displacement in the dual IFM can be solved through the following procedure as

$$[D] = [\overline{D}] + [\Delta D] \quad \text{or} \quad [\Delta D] = [D] - [\overline{D}] \quad (3.57)$$

where $[\overline{D}]$ is the deterministic mean dual matrix.

$$[D]^{-1} = ([\overline{D}] + [\Delta D])^{-1} = ([I] + [A])^{-1}[\overline{D}]^{-1} = ([I] - [A] + [A]^2 - [A]^3 + ...)[\overline{D}]^{-1} \quad (3.58)$$

where $[A] = [\overline{D}]^{-1}[\Delta D]$. Therefore, the displacement in Eq.(3.7) is written as

$$\{X\} = ([I] + [A])^{-1}[\overline{D}]^{-1}\{P^D\}$$

$$= ([I] - [A] + [A]^2 - [A]^3 + ...)[\overline{D}]^{-1}\{P^D\}$$

$$= ([I] - [A] + [A]^2 - [A]^3 + ...)[X_0]$$

$$= \{X_0\} - [A]\{X_0\} + [A]^2\{X_0\} - [A]^3\{X_0\} + ...$$

$$= \{X_0\} - \{X_1\} + \{X_2\} - \{X_3\} + ...$$

$$= \overline{D}[X_i] = [\Delta D][X_{i-1}] \quad i = 1, 2, ... \quad (3.59)$$

It is obvious that the matrix factorization is required only once for all samples in the Neumann expansion technique. Therefore, the computational time can be reduced considerably compared to the direct MCS.

The expansion in Eq.(3.53) and (3.58) will converge if the absolute values of all the eigenvalues of $[A]$ are less than 1.0. The convergence criterion is automatically satisfied, since Gaussian distribution is used for providing the random sample function of all the parameters.
The main formulas of Neumann expansion technique for stochastic analysis in IFM/IFMD are listed in the following Table 3.2.

<table>
<thead>
<tr>
<th>Method</th>
<th>IFM</th>
<th>IFMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>System equations</td>
<td>( {F} = [S]^{-1}{P} )</td>
<td>( {X} = [D]^{-1}{P^D} )</td>
</tr>
<tr>
<td>Series term</td>
<td>( [A] = [S]^{-1}\Delta S )</td>
<td>( [A] = [D]^{-1}\Delta D )</td>
</tr>
<tr>
<td>Recursive equations</td>
<td>( {F_0} = [S]^{-1}{P} )</td>
<td>( {X_0} = [D]^{-1}{P^D} )</td>
</tr>
<tr>
<td>( [S]{F_j} = [\Delta S]{F_{j-1}} )</td>
<td>( [D]{X_j} = [\Delta D]{X_{j-1}} )</td>
<td></td>
</tr>
<tr>
<td>Convergence criterion</td>
<td>( \sum_{k=0}^{i} (-1)^k |F_k|<em>2 \leq \delta</em>{err} )</td>
<td>( \sum_{k=0}^{i} (-1)^k |X_k|<em>2 \leq \delta</em>{err} )</td>
</tr>
</tbody>
</table>

### 3.3 Stochastic sensitivity analysis for IFM/IFMD

Based on stochastic finite element method, attempts are made to exploit the perturbation technique and the Neumann expansion method for the sensitivity analysis of responses obtained by IFM/IFMD in uncertain structures.

This section first highlights the deterministic sensitivity analysis. The perturbation method is used for computation of response stochastic sensitivity analysis within the finite element framework. Finally, the Neumann expansion method with MCS is also utilized to assess the response sensitivity analysis in terms of numerical accuracy and time-efficiency.

#### 3.3.1 Deterministic sensitivity analysis

The governing equation base on IFM can be expressed as

\[
[S\{h\}][F\{h\}] = \{P\{h\}\} \quad (3.61)
\]
where \( \{h\} \) is an array of design variables of size \((M \times 1)\). Note that the governing matrix, internal force vector and load vector in the above equation are the functions of either any design variable in \( \{h\} \) or their combination. For simplicity of analysis, \( \{h\} \) is dropped out in later formulas.

Differentiating Eq.(3.3) with respect to the "\(k\)-th" design variable \(h_k\) results in

\[
[S] \frac{\partial \{F\}}{\partial h_k} + \frac{\partial [S]}{\partial h_k} \{F\} = \frac{\partial \{P\}}{\partial h_k} \quad \text{or} \quad \frac{\partial \{F\}}{\partial h_k} = [S]^{-1} \left( \frac{\partial \{P\}}{\partial h_k} - \frac{\partial [S]}{\partial h_k} \{F\} \right) \tag{3.62}
\]

that is,

\[
\{F_k\} = [S]^{-1} \left( \{P_k\} - \{S_k\} \{F\} \right) \tag{3.63}
\]

where \(\{F_k\}\) is the internal force sensitivity vector with respect to \(h_k\), i.e. \(\{F_k\} = \frac{\partial \{F\}}{\partial h_k}\). \(\{P_k\}\) is the load derivative vector with respect to \(h_k\), i.e. \(\{P_k\} = \frac{\partial \{P\}}{\partial h_k}\). \([S_k]\) is the governing derivative matrix with respect \(h_k\), i.e. \(\frac{\partial [S]}{\partial h_k}\).

Similarly, the deterministic sensitivity formulas of other response in IFM/IFMD can be also obtained in the following Table 3.3.

<table>
<thead>
<tr>
<th>Method</th>
<th>IFM</th>
<th>IFMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/X</td>
<td>({F_k} = [S]^{-1} \left( {P_k} - {S_k} {F} \right))</td>
<td>({X_k} = [D]^{-1} \left( {P_k^D} - {D_k} {x} \right))</td>
</tr>
<tr>
<td>X/F</td>
<td>({X_k} = [J_k] G {F} + { \rho^0 } + [J] [G_k] {F} + [G] {F_k} + { \rho^0_k })</td>
<td>({F_k} = [G]^{-1} \left( [B]^T {X_k} - { \rho^0 } - [G_k] {F} \right))</td>
</tr>
<tr>
<td>Stress</td>
<td>({\sigma_k} = [Y_k] {F} + {Y} {F_k})</td>
<td>({\sigma_k} = [Y_k] {F} + {Y} {F_k})</td>
</tr>
<tr>
<td>Strain</td>
<td>({\varepsilon_k} = [Z_k] {X} + {Z} {X_k})</td>
<td>({\varepsilon_k} = [Z_k] {X} + {Z} {X_k})</td>
</tr>
</tbody>
</table>

It is noted that the deformation coefficient matrix \([J]\) is the function of \(h_k\), but the equilibrium matrix \([B]\) is not the function of \(h_k\). Furthermore, the deterministic
sensitivity formulas of IFM/IFMD are more complicated than the deterministic analysis ones.

3.3.2 Stochastic sensitivity analysis in IFM/IFMD

It is well versed in SFEM that the uncertain structure parameters cause the responses uncertain. As a result, the sensitivity will no longer remain deterministic. Likewise, the perturbation technique can be also employed in the stochastic sensitivity analysis of the structural response. Therefore, the variation of matrix \([S_k]\) and vectors \(\{F_k\}\) and \(\{P_k\}\) can be expressed in the form of a second-order Taylor series expansion in relation to the random variables. Note that the expansions of the governing matrix \([S]\) and the internal force vector \(\{F\}\) are already performed in Eq.(3.23) and Eq.(3.21).

\[
\{F_k\} = \overline{\{F_k\}} + \sum_{i=1}^{N} \{F_{k,i}\} q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_{k,ij}\} q_i q_j + \ldots \quad (3.64a)
\]

\[
\{P_k\} = \overline{\{P_k\}} + \sum_{i=1}^{N} \{P_{k,i}\} q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{P_{k,ij}\} q_i q_j + \ldots \quad (3.64b)
\]

\[
[S_k] = \overline{[S_k]} + \sum_{i=1}^{N} [S_{k,i}] q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} [S_{k,ij}] q_i q_j + \ldots \quad (3.64c)
\]

where \(\overline{\{F_k\}}, \overline{\{P_k\}}\) and \(\overline{[S_k]}\) denote the corresponding deterministic part of vector or matrix at \(q_i = 0\) \((i = 1, 2, \ldots, N)\), other coefficients are defined as follows

\[
\{F_{k,ij}\} = \frac{\partial \{F_k\}}{\partial q_i} \bigg|_{\{q\} = \{0\}} \quad \{F_{k,ij}\} = \frac{\partial^2 \{F_k\}}{\partial q_i \partial q_j} \bigg|_{\{q\} = \{0\}}
\]
Substituting Eq. (3.21), (3.23) and (3.64) into Eq. (3.63) and equating equal order terms, the zeroth-, first- and second-order equations corresponding to Eq. (3.63) are

Zeroth-order:

\[
[F_k] = [S]^{-1}([P_k] - [S_k][F])
\]  
(3.66a)

First-order:

\[
[F_{k,j}] = [S]^{-1}([P_{k,j}] - [S_{k,j}][F_j] - [S_{k,j}][F] - [S_{j}] [F_k])
\]  
(3.66b)

Second-order:

\[
[F_{k,j}] = [S]^{-1}([P_{k,j}] - [S_{k,j}][F_j] - 2[S_{k,j}] [F_j] - [S_{j}] [F_{k,j}] - 2[S_{j}] [F_{k,j}] - [S_{j}] [F_{k}])
\]  
(3.66c)

Finally, the first-order approximation for the internal force sensitivity is obtained by

\[
\{F_k\} = \{\overline{F}_k\} + \sum_{i=1}^{N} \{F_{k,i}\} q_i
\]

\[
\mu'_{\overline{F}_k} = E' \{\{F_k\}\} = \{\overline{F}_k\}
\]

\[
Cov' \{\{F_k\}, \{F_k\}^T\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_{k,i}\} \{F_{k,j}\}^T \text{E}[q_i q_j]
\]  
(3.67)

and the second-order approximation is

\[
\{F_k\} = \{\overline{F}_k\} + \sum_{i=1}^{N} \{F_{k,i}\} q_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{F_{k,i,j}\} q_i q_j
\]
\[
\mu^H_{F_k} = E^H [\{F_k\}] = (\bar{F}_k) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} E[q_i q_j]
\]

\[
\text{Cov}^H(\{F_k\}, \{F_k\}^T) = \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} \{F_{k,ij}\}^T E[q_i q_j]
\]  \hspace{1cm} (3.68)

Similarly, other response sensitivity formulas of IFM/IFMD can be also obtained in the following Table 3.4.

### Table 3.4 Stochastic sensitivity formulas in IFM/IFMD

#### IFM

\[
\begin{align*}
[F_k] &= \left[ \begin{array}{c} \bar{F}_k \\ F_j \\ \end{array} \right] \\
\{F_{k,ij}\} &= \left[ \begin{array}{c} \{F_{k,ij}\} \\ \end{array} \right] \\
[\{F_{k,ij}\}^T] &= \left[ \begin{array}{c} \{F_{k,ij}\}^T \\ \end{array} \right]
\end{align*}
\]

#### IFMD

\[
\begin{align*}
[\bar{X}_k] &= \left[ \begin{array}{c} \bar{G} \\ \end{array} \right] \\
\{X_{k,ij}\} &= \left[ \begin{array}{c} \{X_{k,ij}\} \\ \end{array} \right]
\end{align*}
\]

<table>
<thead>
<tr>
<th>The first-order approximation</th>
<th>The second-order approximation</th>
</tr>
</thead>
</table>
| \[
\begin{align*}
\{F_k\} &= \left( \begin{array}{c} \bar{F}_k \\ \end{array} \right) + \sum_{i=1}^N \{F_{k,ij}\} q_i \\
\mu^I_{F_k} &= E^I [\{F_k\}] = \left( \begin{array}{c} \bar{F}_k \\ \end{array} \right) \\
\text{Cov}^I(\{F_k\}, \{F_k\}^T) &= \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} \{F_{k,ij}\}^T E[q_i q_j]
\end{align*}
\] | \[
\begin{align*}
\{F_k\} &= \left( \begin{array}{c} \bar{F}_k \\ \end{array} \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} q_i q_j \\
\mu^H_{F_k} &= E^H [\{F_k\}] = \left( \begin{array}{c} \bar{F}_k \\ \end{array} \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} q_i q_j \\
\text{Cov}^H(\{F_k\}, \{F_k\}^T) &= \sum_{i=1}^N \sum_{j=1}^N \{F_{k,ij}\} \{F_{k,ij}\}^T E[q_i q_j]
\end{align*}
\] |
Table 3.4 Stochastic sensitivity formulas in IFM/IFMD (Continued)

<table>
<thead>
<tr>
<th>The first-order approximation</th>
<th>The second-order approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {X_k} = {\bar{X}<em>k} + \sum</em>{i=1}^{N} {x_{k,i}} \tilde{a}_i )</td>
<td>( {X_k} = {\bar{X}<em>k} + \sum</em>{i=1}^{N} {x_{k,i}} \tilde{a}<em>i + \frac{1}{2} \sum</em>{i=1}^{N} \sum_{j=1}^{N} {x_{k,ij}} \tilde{a}_i \tilde{a}_j )</td>
</tr>
<tr>
<td>( \mu_{X_k} = E'[{X_k}] = {\bar{X}_k} )</td>
<td>( \mu_{X_k} = E''[{X_k}] = {\bar{X}<em>k} + \frac{1}{2} \sum</em>{i=1}^{N} \sum_{j=1}^{N} {x_{k,ij}} E[\tilde{a}_i \tilde{a}_j] )</td>
</tr>
<tr>
<td>( \text{Cov}''({X_k}, {X_k}^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} {x_{k,ij}} {x_{k,ij}}^T E[\tilde{a}_i \tilde{a}_j] )</td>
<td>( \text{Cov}''({X_k}, {X_k}^T) = \sum_{i=1}^{N} \sum_{j=1}^{N} {x_{k,ij}} {x_{k,ij}}^T E[\tilde{a}_i \tilde{a}_j] )</td>
</tr>
</tbody>
</table>

Stress

\[
\begin{align*}
\{\sigma_k\} & = \{\bar{\sigma}_k\} + \sum_{i=1}^{N} \{\sigma_{k,i}\} \tilde{a}_i \\
\mu_{\sigma_k} & = E'[\{\sigma_k\}] = \{\bar{\sigma}_k\} \\
\text{Cov}''(\{\sigma_k\}, \{\sigma_k\}^T) & = \sum_{i=1}^{N} \sum_{j=1}^{N} \{\sigma_{k,ij}\} \{\sigma_{k,ij}\}^T E[\tilde{a}_i \tilde{a}_j] \\
\end{align*}
\]

Strain

\[
\begin{align*}
\{\varepsilon_k\} & = \{\bar{\varepsilon}_k\} + \sum_{i=1}^{N} \{\varepsilon_{k,i}\} \tilde{a}_i \\
\mu_{\varepsilon_k} & = E'[\{\varepsilon_k\}] = \{\bar{\varepsilon}_k\} \\
\text{Cov}''(\{\varepsilon_k\}, \{\varepsilon_k\}^T) & = \sum_{i=1}^{N} \sum_{j=1}^{N} \{\varepsilon_{k,ij}\} \{\varepsilon_{k,ij}\}^T E[\tilde{a}_i \tilde{a}_j] \\
\end{align*}
\]

Note: \( k \) is the number of design variables, \( k = 1, 2, \ldots, M \).

It is clearly observed that the stochastic sensitivity formulas have similar features of the stochastic analysis formulas. The zeroth-order equations are identical to the deterministic sensitivity equations evaluated at the mean value. The first- and second-order equations consist of the recursive items. Moreover, the first- or second-order
approximation formulas of IFM/IFMD have the same forms in representation. The formulas of stochastic sensitivity analysis of IFM/IFMD are more sophisticated.

Based on the reliability-based method, stochastic sensitivity analysis formulation may also be expressed by another form. Since the first two probabilistic moments of responses have been obtained in the previous section, the response value at p probability of occurrence can be calculated by the inverse of the cumulative distribution function.

Let \( v \), \( \mu_v \) and \( \sigma_v \) denote the response value of p probability of occurrence, the mean value and standard deviation, respectively. By means of the transformation, the inverse of the cumulative distribution function can be written as

\[
\Phi^{-1}(p) = \frac{v - \mu_v}{\sigma_v}
\]

i.e.

\[
v = \mu_v + \Phi^{-1}(p)\sigma_v
\]

where \( \Phi^{-1}(p) \) is the cumulative distribution function for the standard normal distribution.

The sensitivity of \( v \) with respect to the mean of a primitive random variable, \( \mu_R \) can be given by

\[
\frac{\partial v}{\partial \mu_R} = \frac{\partial \mu_v}{\partial \mu_R} + \Phi^{-1}(p) \frac{\partial \sigma_v}{\partial \mu_R}
\]

It should be noted that the response sensitivity is the sum of the derivatives of the mean response and the standard deviation that is prorated by the inverse standard normal cumulative distribution function \( \Phi^{-1}(p) \). Obviously, the sensitivity formula is a complicated expression, even for the simple structure. It is elevated in the closed form for 15 examples in Chapter V, by using Maple software.
3.3.3 Neumann expansion for stochastic sensitivity analysis

The Neumann expansion with Monte Carlo simulation now can also be extended for computation of response sensitivity in IFM/IFMD. Setting \( \{p_{k}^*\} = \left( \frac{\partial [P]}{\partial h_k} - \frac{\partial [S]}{\partial h_k} \{F\} \right) \) in Eq.(3.64), the original Eq.(3.64) can be written as

\[
\{F_k\} = [S]^{-1} \{p_{k}^*\}
\]

To obtain the sensitivity formulas of response with respect to \( h_k \). Substituting Eq.(3.53) into Eq.(3.71) yields

\[
\{F_k\} = [S]^{-1} \{p_{k}^*\} \\
= \left( [I] - [A]^T - [A]^T + ... [S]^{-1} \{p_{k}^*\} \right) \\
= \left( [I] - [A]^T - [A]^T + ... [F_k^0] \right) \\
= \{F_{k0}\} - [A] \{F_{k1}\} + [A]^T \{F_{k2}\} - [A]^T \{F_{k3}\} + ...
\]

where \([A] = [S]^{-1} \{\Delta S\}\). \( \{F_{k0}\} \) is the deterministic part, \( \{F_{k0}\} = [S]^{-1} \{p_{k}^*\} \) and other terms of expansion are the fluctuating components of the random sensitivity. The expansion series can also be expressed by the recursive equation as

\[
[S]\{F_{ki}\} = [\Delta S]\{F_{k(i-1)}\}
\]

It is evident that the computation of \( \{F_{k0}\} \) in each simulation does not involve any further decomposition of \([S]\). Once \( \{F_{k0}\} \) is obtained by using \( \{F_{k0}\} = [S]^{-1} \{p_{k}^*\} \), the random sensitivity of internal force with respect to \( h_k \) can be iteratively calculated for each simulation.

Analogous to the stochastic response analysis in section 3.2.3, the expansion series in Eq.(3.72) may be terminated after a few terms if convergence criterion is used as...
\[
\left\| \{F_{ki}\} \right\|_2 \leq \delta_{err}
\]

\[
\sum_{j=0}^{\infty} (-1)^j \left\| \{F_{kj}\} \right\|_2 \leq \delta_{err}
\]

where \(\delta_{err}\) is the allowable error, \(\delta_{err} = 0.01\).

Other stochastic sensitivity of responses such as \(\{X_k\}\), \(\{\sigma_k\}\) and \(\{e_k\}\) in the IFM can be obtained through equations in Table 3.2 and the obtained results \(\{F_k\}\) in each simulation.

Similarly, the stochastic sensitivity formulas in the dual IFM can be solved through the following expansions:

\[
\{p_k^{D*}\} = \left( \frac{\partial [P^0]}{\partial h_k} - \frac{\partial [D]}{\partial h_k} \right) \{X\}
\]

\[
\{X_k\} = [D]^{-1} \{p_k^{D*}\}
\]

\[
= \left( [I] - [A] + [A]^2 - [A]^3 + \ldots \right) [D]^{-1} \{p_k^{D*}\}
\]

\[
= \left( [I] - [A] + [A]^2 - [A]^3 + \ldots \right) \{X^0\}
\]

\[
= \{X_{k_0}\} - \{X_{k_1}\} + \{X_{k_2}\} - \{X_{k_3}\} + \ldots
\]

\[
[D] = [D]^{-1} \{\Delta D\}
\]

\[
\{X_{k_0}\} = [D]^{-1} \{p_k^{D*}\}
\]

\[
[D] \{X_{ki}\} = [\Delta D] \{X_{k(i-1)}\}
\]

It should be noted that the stochastic sensitivity analysis by NE-MCS technique has the same features as the stochastic response analysis such as reducing computation time, convergence criterion, and convergence. The main formulas of Neumann expansion technique for stochastic sensitivity analysis in IFM/IFMD are shown in the following Table 3.5.
Table 3.5 Neumann Expansion for stochastic analysis formulation in IFM/IFMD

<table>
<thead>
<tr>
<th>Method</th>
<th>IFM</th>
<th>IFMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>System equations</td>
<td>${F_k} = [S]^{-1} [F_k^*]$</td>
<td>${X} = [D]^{-1} [P^D]$</td>
</tr>
<tr>
<td></td>
<td>${F_k^*} = \left( \frac{\partial [P]}{\partial h_k} - \frac{\partial [S]}{\partial h_k} \right) [F_k]$</td>
<td>${P^D} = \left( \frac{\partial [P^D]}{\partial h_k} - \frac{\partial [D]}{\partial h_k} \right) [X]$</td>
</tr>
<tr>
<td>Series term</td>
<td>$[A] = [S]^{-1} [\Delta S]$</td>
<td>$[A] = [D]^{-1} [\Delta D]$</td>
</tr>
<tr>
<td>Recursive equations</td>
<td>${F_{k0}} = [S]^{-1} [F_k^*]$</td>
<td>${X_{k0}} = [D]^{-1} [P^D]$</td>
</tr>
<tr>
<td></td>
<td>$[S] [F_{ki}] = [\Delta S] [F_{k(i-1)}]$</td>
<td>$[D] [X_{ki}] = [\Delta D] [X_{k(i-1)}]$</td>
</tr>
<tr>
<td>Convergence criterion</td>
<td>$\left| F_{ki} \right|<em>2 \leq \delta</em>{err}$</td>
<td>$\left| X_{ki} \right|<em>2 \leq \delta</em>{err}$</td>
</tr>
</tbody>
</table>

3.4 Stochastic variational principles of IFM

To find an exact analytical solution to the uncertain structure problems, an alternative formulation, referred to as the variational formulation, has been extensively investigated in monographs [89-92]. In fact, the variational formulation is considered physically to be the only natural and rigorously correct way. It is noted that the variational statement concentrate all of the intrinsic features of problem in a single functional.

Neglecting theoretical details of variational formulations of IFM, let us now illustrate the generalization of the stochastic variational principles for the inclusion of the random effecting by using the second-order perturbation technique.

Assume that only linear static deformation processes are considered in this section. Therefore, the variational functional in IFM, Eq.(3.3), can be expressed in indicial notation as
\[ \pi_s = \int_{\Omega} \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \, d\Omega + \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega - \int_{\partial \Omega} P_i \vec{n}_i \, d(\partial \Omega) - \int_{\partial \Omega} \vec{F}_i u_i \, d(\partial \Omega) - \int_{\Omega} B_i u_i \, d\Omega \quad i,j=1,2,3 \quad (3.79) \]

where the strain \( \varepsilon_{ij} \) is a function of \( u_i \) according to

\[ \varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \quad x \in \Omega \quad (3.80) \]

The stresses \( \sigma_{ij} \) are described by means of the symmetric stress tensor \( \sigma = \frac{1}{2} \sigma_{ij} \). They are related to strains by means of the constitutive relations, Hooke’s law, as

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad x \in \Omega \quad (3.81) \]

\( B_i \) is the body force in terms of the equations of motion

\[ \sigma_{ij} + B_i = 0 \quad x_k \in \Omega \quad (3.82) \]

\( u_i \) is the displacement satisfied the boundary condition

\[ u_i = \hat{u}_i \quad x_k \in \partial \Omega_u \quad (3.83) \]

\( P_i \) is the tractions in terms of the boundary condition

\[ \sigma_{ij} n_j = P_i \quad x_k \in \partial \Omega_{\sigma} \quad (3.84) \]

where \( n_j \) is the unit vector normal to the boundary \( \partial \Omega \). \( \partial \Omega_u \cup \partial \Omega_{\sigma} = \partial \Omega \) and \( \partial \Omega_u \cup \partial \Omega_{\sigma} = \phi \)

In the above equations, \( u_i \) are the components of displacements, \( x_i \) are the spatial coordinates, \( C_{ijkl} \) are the components of the material response tensor. Repeated indices denote sum and a comma denotes the partial differentiation.

Substituting Eq.(3.81) into Eq.(3.79) yields

\[ \pi_s = \int_{\Omega} \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, d\Omega + \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega - \int_{\partial \Omega} P_i \vec{n}_i \, d(\partial \Omega) - \int_{\partial \Omega} \vec{F}_i u_i \, d(\partial \Omega) - \int_{\Omega} B_i u_i \, d\Omega \quad (3.85) \]
By using the symmetry of the tensor $C_{ijkl}$ and Eq.(3.80) to express $\varepsilon_{ij}$ in terms of $u_i$, we obtain the stationary condition of the variational function as

$$\delta\pi_s = \int_{\Omega} \left[ \frac{1}{2} C_{ijkl} u_{i,j} \delta u_{k,l} d\Omega + \int_{\Omega} \sigma_{ij}^c \delta u_{i,j} d\Omega - \int_{\Omega} P_i \delta \bar{u}_i d(\partial\Omega) - \int_{\Omega} \bar{P}_i \delta u_i d(\partial\Omega) - \int_{\Omega} B_i \delta u_i d\Omega \right] = 0$$

or

$$\int_{\Omega} \left[ \frac{1}{2} C_{ijkl} u_{i,j} \delta u_{k,l} d\Omega + \int_{\Omega} \sigma_{ij}^c \delta u_{i,j} d\Omega = \int_{\Omega} P_i \delta \bar{u}_i d(\partial\Omega) + \int_{\Omega} \bar{P}_i \delta u_i d(\partial\Omega) + \int_{\Omega} B_i \delta u_i d\Omega \right] = 0 \quad (3.86)$$

Assume a set of $R$ random fields, $\{b(x_k)\}$, as

$$b(x_k) = \{b(x_1), b(x_2), \ldots, b(x_k)\} \quad k = 1, 2, 3, \ldots \quad (3.87)$$

which can represent randomness in structural parameters. The expected values of their random fields, $b_r(x_k), r = 1, 2, \ldots, R$, are defined as

$$E\{b_r\} = b_r^0 = \int_{-\infty}^{\infty} b_r p_1(b_r) db_r \quad (3.88a)$$

$$\text{cov}(b_r, b_s) = S_{bs}^r = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (b_r - b_r^0)(b_s - b_s^0) p_2(b_r, b_s) db_r db_s \quad r, s = 1, 2, \ldots, R \quad (3.88b)$$

where $p_1(b_r)$ and $p_2(b_r, b_s)$ denote the probability density function and the joint probability density function, respectively.

In accordance with the philosophy of the perturbation approach, all the random variables in the Eq.(3.88b) should be first expanded about the spatial expectations of the random field variables $b(x_k) = \{b_r(x_k)\}$, via Taylor series expansions with a given small parameter $\varepsilon$ and retaining terms up to the second order. The expansions can be explicitly expressed as
\[ C_{ijkl}[b(x_k); x_k] = C_{ijkl}^0[b^0(x_k); x_k] + 3 C_{ijkl}^r[b^0(x_k); x_k] \Delta b_r(x_k) + \frac{1}{2} \varepsilon^2 C_{ijkl}^{rr}[b^0(x_k); x_k] \Delta b_r(x_k) \Delta b_r(x_k) \]

\[ \sigma^2_{ij}[b(x_k); x_k] = \sigma^2_{ij}^0[b^0(x_k); x_k] + 3 \sigma^2_{ij}^r[b^0(x_k); x_k] \Delta b_r(x_k) + \frac{1}{2} \varepsilon^2 \sigma^2_{ij}^{rr}[b^0(x_k); x_k] \Delta b_r(x_k) \Delta b_r(x_k) \]

\[ P_i[b(x_k); x_k] = P_i^0[b^0(x_k); x_k] + 3 P_i^r[b^0(x_k); x_k] \Delta b_r(x_k) + \frac{1}{2} \varepsilon^2 P_i^{rr}[b^0(x_k); x_k] \Delta b_r(x_k) \Delta b_r(x_k) \]

\[ B_i[b(x_k); x_k] = B_i^0[b^0(x_k); x_k] + 3 B_i^r[b^0(x_k); x_k] \Delta b_r(x_k) + \frac{1}{2} \varepsilon^2 B_i^{rr}[b^0(x_k); x_k] \Delta b_r(x_k) \Delta b_r(x_k) \]

\[ u_i[b(x_k); x_k] = u_i^0[b^0(x_k); x_k] + 3 u_i^r[b^0(x_k); x_k] \Delta b_r(x_k) + \frac{1}{2} \varepsilon^2 u_i^{rr}[b^0(x_k); x_k] \Delta b_r(x_k) \Delta b_r(x_k) \]

(3.89)

where \( b^0(x_k) = \{b^0_r(x_k)\} \) denotes the spatial expectations.

\[ \varepsilon \Delta b_r(x_k) = \partial b_r(x_k) = \varepsilon [b_r(x_k) - b^0_r(x_k)] \]

(3.90)

is the first-order variation of \( b_r(x_k) \) about \( b^0_r(x_k) \) and

\[ \varepsilon^2 \Delta b_r(x_k) \Delta b_r(x_k) = \partial b_r(x_k) \partial b_r(x_k) = \varepsilon^2 [b_r(x_k) - b^0_r(x_k)][b_r(x_k) - b^0_r(x_k)] \]

(3.91)

represent the second-order variation of \( b_r(x_k) \) and \( b^0_r(x_k) \) about \( b^0_r(x_k) \) and \( b^0_r(x_k) \), respectively. The notation \( (\cdot)^0 \) indicates the expected value, whereas \( (\cdot)^r \) and \( (\cdot)^{rr} \) indicated the first and second partial derivatives with respect to the random field variables \( b_r(x_k) \) evaluated at their expectations, respectively.

Then, substituting the expansions above into Eq.(3.86) and equating the components of the same order, the zeroth-, first- and second-order variational statements are obtained as

Zeroth-order (\( \varepsilon^0 \) terms, one equation)
\[
\int C_{i,j}^0 u_{i,j}^0 \delta u_{i,j} d\Omega = \int P_i^0 \delta u_i d(\partial \Omega) + \int P_i^0 \delta u_i d(\Delta \Omega) + \int B_i^0 \delta u_i d\Omega - \int \sigma_{ij}^{0} \delta u_{i,j} d\Omega
\]  
(3.92)

First-order (3 terms, R equations)
\[
\int C_{i,j}^0 u_{i,j}^0 \delta u_{i,j} d\Omega = \int P_i^0 \delta u_i d(\partial \Omega) + \int P_i^0 \delta u_i d(\Delta \Omega) + \int B_i^0 \delta u_i d\Omega - \int \sigma_{ij}^{0} \delta u_{i,j} d\Omega
\]
(3.93)

Second-order (3 terms, one equation)
\[
\int C_{i,j}^0 u_{i,j}^{rs} \delta u_{i,j} d\Omega = \int P_i^{rs} S_k^{rs} \delta u_i d(\partial \Omega) + \int P_i^{rs} S_k^{rs} \delta u_i d(\Delta \Omega) + \int B_i^{rs} S_k^{rs} \delta u_i d\Omega
\]
(3.94)

It should be noticed that the test function \(\delta u_i\) should satisfy the kinematic boundary conditions, Eq.(3.83) on \(\partial \Omega\). Furthermore, the second-order equation is obtained above by multiplying the R-variate probability density function \(p_R(b_1, b_2, ... b_k)\) by the 3 terms and integrating over random vector \(b(x)\) domain. For example, the third term in the left-hand side of Eq.(3.94) holds
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_i^{rs} \left[ B_0^0(x_k) \Delta b_r(x_k) \Delta b_s(x_k) \delta u_i d(\Omega) \right] p_R(b(x_k))db
\]
(3.95)

Thus, it can be observed that the occurrence of the double sums \(\sum \sum S_k^{rs}\) and \(\sum (\cdot)^{rs} S_k^{rs}\) in the formulation makes it possible to represent an equation of the second-order, instead of \((R+1)R/2\) analogous equations.
Eq.(3.92) is seen to be identical to the deterministic expression of the stationary condition of the variational functional. Thus, it can be solved for \( u_i^0(x_k) \). Then, the first- and second-order equations can be evaluated, sequentially, for \( u_i^{r\gamma}(x_k) \) and \( u_i^{r\gamma\gamma}(x_k) \).

In search of the probabilistic distributions of the displacements, setting \( \varepsilon = 0 \) yields the deterministic solution, whereas setting \( \varepsilon = 1 \) stipulates that the fluctuation of the random field variables \( b(x_k) \) is small. Therefore, taking the expected value and covariance of the expanded equation in the random displacement field \( u_i[b(x_k);x_k] \) of Eq.(3.89) yields

\[
E[u_i[b(x_k);x_k]] = \int_{-\infty}^{\infty} u_i[b(x_k);x_k] P_R(b(x_k)) db
\]

\[
= \int_{-\infty}^{\infty} u_i^0[b^0(x_k);x_k] + u_i^{r\gamma}[b^0(x_k);x_k] \Delta b_r(x_k) + \left( \frac{1}{2} u_i^{r\gamma\gamma}[b^0(x_k);x_k] \Delta b_r(x_k) \Delta b_s(x_k) \right) P_R(b(x_k)) db
\]

\[
= u_i^0 \int_{-\infty}^{\infty} P_R(b(x_k)) db + u_i^{r\gamma} \int_{-\infty}^{\infty} \Delta b_r(x_k) P_R(b(x_k)) db + \frac{1}{2} u_i^{r\gamma\gamma} \int_{-\infty}^{\infty} \Delta b_r(x_k) \Delta b_s(x_k) P_R(b(x_k)) db
\]

\[
= u_i^0 + \frac{1}{2} u_i^{r\gamma\gamma} S^\gamma_b
\]

(3.96)

Cov\((u_i[b(x_k^{(1)});x_k^{(1)}], u_j[b(x_k^{(2)});x_k^{(2)}])\)

\[
= S^\gamma_{ij}(x_k^{(1)}, x_k^{(2)})
\]

\[
= \int_{-\infty}^{\infty} \left[ u_i[b(x_k^{(1)});x_k] - E[u_i[b(x_k^{(1)});x_k]] \right] \left[ u_j[b(x_k^{(2)});x_k] - E[u_j[b(x_k^{(2)});x_k]] \right] P_R(b(x_k)) db
\]

(3.97)

Thus,

\[
S^\gamma_{ij}(x_k^{(1)}, x_k^{(2)}) = u_i^{r\gamma}(x_k^{(1)}) u_j^{r\gamma}(x_k^{(1)}) S^\gamma_b
\]

(3.98)

where \( \varepsilon = 1 \) is applied. Note that the formulas above are the second order accuracy of mean value and covariance of the displacement estimation. Clearly, the first-order estimate of mean value is obtained as
\[ E[u_i[b(x_k),x_k]] = u_i^0(x_k) \] (3.99)

and the first-order accuracy of covariance is consistent with the second-moment analysis.

To determine the first two probabilistic moments of the strains, using Eq.(3.64) yields

The second-order accurate mean value

\[
E[\varepsilon_{ij}[b(x_k),x_k]] = E\left[\frac{1}{2}(u_{i,j}^0[b(x_k),x_k] + u_{j,i}^0[b(x_k),x_k])\right] = \frac{1}{2}E\left[ u_{i,j}^0(x_k) + u_{j,i}^0(x_k) + \frac{1}{2}(u_{i,j}^{rs}(x_k) + u_{j,i}^{rs}(x_k))S_{rs}^b \right]
\]

\[ = \varepsilon_{ij}^0(x_k) + \frac{1}{2}\varepsilon_{ij}^{rs}(x_k)S_{rs}^b \] (3.100)

The second-order accurate covariance

\[
\text{Cov}(\varepsilon_{ij}[b(x_k^{(1)},x_k^{(1)})],\varepsilon_{lm}[b(x_k^{(2)},x_k^{(2)})]) = S_{ijlm}^{rs}(x_k^{(1)},x_k^{(2)}) = \frac{1}{4}[ u_{i,j}^{rs}(x_k^{(1)}) + u_{j,i}^{rs}(x_k^{(1)})u_{j,m}^{rs}(x_k^{(2)}) + u_{m,j}^{rs}(x_k^{(2)})]S_{rs}^b
\]

\[ = \varepsilon_{ij}^{rs}(x_k^{(1)})\varepsilon_{lm}^{rs}(x_k^{(2)})S_{rs}^b \] (3.101)

in which the following expansion has been employed.

\[
\varepsilon_{ij}[b(x_k),x_k] = \varepsilon_{ij}^{0}[b^0(x_k),x_k] + \varepsilon_{ij}^{\Delta}[b^0(x_k),x_k]\Delta b_i(x_k) + \frac{1}{2}\varepsilon_{ij}^{rs}[b^0(x_k),x_k]\Delta b_i(x_k)\Delta b_j(x_k)
\]

(3.102)

The first-order accurate mean value

\[ E[\varepsilon_{ij}[b(x_k),x_k]] = \varepsilon_{ij}^0(x_k) = \frac{1}{2}E[u_{i,j}^0(x_k) + u_{j,i}^0(x_k)] \] (3.103)

and the first-order accurate covariance is consistent with the second-order accurate covariance.
Substituting the expansions of $C_{ijkl}[b(x_k),x_k]$ in Eq.(3.89) and $\varepsilon_{ij}[b(x_k),x_k]$ in Eq.(3.102) into Eq.(3.103), we obtain

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$= \left( C_{ijkl}^{0} + C_{ijkl}^{m} \Delta b_{r} + \frac{1}{2} C_{ijkl}^{mm} \Delta b_{r} \Delta b_{w} \right) \left( \varepsilon_{ij}^{0} + \varepsilon_{ij}^{m} \Delta b_{r} + \frac{1}{2} \varepsilon_{ij}^{mm} \Delta b_{r} \Delta b_{w} \right)$$

(3.104)

Analogously to Eq.(3.96) and Eq.(3.97) and neglecting the variations of the terms higher than two-order, we arrive at:

The second-order accurate mean value:

$$E\left[ \sigma_{ij} \left[ b(x_k),x_k \right] \right] = C_{ijkl}^{0} \left( x_k \right) \varepsilon_{kl}^{0} \left( x_k \right) + \frac{1}{2} \left[ C_{ijkl}^{m} \left( x_k \right) \varepsilon_{kl}^{m} \left( x_k \right) + 2 C_{ijkl}^{r} \left( x_k \right) \varepsilon_{kl}^{r} \left( x_k \right) + C_{ijkl}^{0} \left( x_k \right) \varepsilon_{kl}^{0} \left( x_k \right) \right]$$

$$S_{b}^{\sigma}$$

(3.105)

The second-order accurate covariance

$$Cov\left( \sigma_{ij} \left[ b(x^{(1)}_k),x^{(1)}_k \right] \sigma_{lm} \left[ b(x^{(2)}_k),x^{(2)}_k \right] \right)$$

$$= S_{b}^{\sigma}$$

$$= \left[ C_{ijkl}^{r} \left( x^{(1)}_k \right) C^{sr}_{lm} \left( x^{(2)}_k \right) \varepsilon_{pq}^{0} \left( x^{(1)}_k \right) \varepsilon_{pq}^{0} \left( x^{(2)}_k \right) + C_{ijkl}^{r} \left( x^{(1)}_k \right) C^{sr}_{lm} \left( x^{(2)}_k \right) \varepsilon_{pq}^{0} \left( x^{(1)}_k \right) \varepsilon_{pq}^{0} \left( x^{(2)}_k \right) + C_{ijkl}^{r} \left( x^{(1)}_k \right) C^{sr}_{lm} \left( x^{(2)}_k \right) \varepsilon_{pq}^{0} \left( x^{(1)}_k \right) \varepsilon_{pq}^{0} \left( x^{(2)}_k \right) \right] S_{b}^{\sigma}$$

(3.106)

It should be underlined that the method of the variational formulation has quite a general character and randomizes any differential or algebraic equations with respect to the parameters by using the second-order perturbation technique as well as obtaining the second moments of all the states variables.
In engineering problems, randomness and uncertainties in structural parameters are inherent and frequently lead to a key difficulty. The difficulty prohibits the application of the current design optimization methods, because of the extensive computation in both approximating the expected value of the performance function and setting up the expressions of constraints. With aid of SFEM, the stochastic optimization solves the problem with large-scale and highly nonlinear models.

CometBoards is a multidisciplinary design optimization test-bed. The code formulates design as a nonlinear mathematical programming problem and solves it. CometBoards has been successfully applied for structural design optimization, subsonic and supersonic aircraft problems and different types of jet engine problems. Through the perturbation technique, the stochastic responses are completely described by the stochastic nature of the design variables. The merit function and stochastic constraints are also obtained. The probabilistic design problem is to find the mean values of the design parameters that optimize the mean value of a merit function (such as weight) subjected to a set of stochastic constraints. The stochastic optimal design problem is solved using the sequential quadratic programming method (SQP) in this study.
To make the system performance insensitive to the variations of the design variables, robust design optimization of structures under uncertainty is investigated. The best design, therefore, may be chosen by a trade-off decision between optimization of the mean and minimization of the variance of the performance.

In the following sections, firstly, the merit function and constraint formulation for the stochastic design are described. Secondly, sequential quadratic programming method in CometBoards is introduced and formulations of stochastic optimization are provided. Finally, formulations of robust design optimization are presented.

4.1 Merit function and constraint formulation for stochastic design

The optimum evaluated from deterministic optimization may violate the imposed constraints or cause the system performance defined as the objective function to be varied drastically. They are generated from the uncertainties on structural parameters. As a result, the merit function and constraint formulation need to be considered through the stochastic approaches.

4.1.1 Weight as the merit function

Weight is often used as the merit function in design optimization. The weight function $W$ in deterministic form of structure can be written as

$$W = \sum_{k=1}^{n} \rho_{k} A_{k} l_{k}$$  \hspace{1cm} (4.1)

where $\rho_{k}$, $A_{k}$ and $l_{k}$ are the weight density, the cross-sectional area and the length of the $k$-th element, respectively.
As mentioned in 3.2.1, the cross-sectional area $A_k$ and the weight density $\rho_k$ should be treated as the random variables due to the usual manufacturing tolerances and material property variation, respectively. The random variables are identified as two-dimensional, univariate, homogeneous stochastic fields, and can be modeled as

$$A_k(x) = \mu_{A_k} (1 + q_{A_k}(x))$$

$$\rho_k(x) = \mu_{\rho_k} (1 + q_{\rho_k}(x))$$

(4.2)

Where $\mu_{A_k}$ and $\mu_{\rho_k}$ are the expected values of $A_k$ and $\rho_k$, $x = [x, y]^T$ denotes the position vector. $q_{A_k}(x)$ and $q_{\rho_k}(x)$ are two independent univariate real homogeneous stochastic fields with zero-mean. Here, the length, $l_k$, of the $k$-th element is assumed to be deterministic.

Thus, the weight function $W$ in stochastic form may be obtained by

$$W = \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k (1 + q_{\rho_k}(x))(1 + q_{A_k}(x))$$

(4.3)

and taking the expectation of Eq.(4.3) yields

$$\mu_W = E[W] = \sum_{k=1}^{n} E[\mu_{\rho_k} \mu_{A_k} l_k (1 + q_{\rho_k}(x))(1 + q_{A_k}(x))]$$

$$= \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k E[(1 + q_{\rho_k}(x))(1 + q_{A_k}(x))]$$

(4.4)

$$= \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k (1 + E[q_{\rho_k}(x)q_{A_k}(x)])$$

$$= \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k + \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k \text{cov}(q_{\rho_k}(x), q_{A_k}(x))$$

Note that the covariance of $q_{A_k}(x)$ and $q_{\rho_k}(x)$ is zero, $\text{Cov}(q_{\rho_k}, q_{A_k}) = 0$ due to their independence.

Thus,
\[
\mu_W = E[W] = \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k
\]  
(4.5)

and the covariance of \( W \) can be obtained by

\[
Cov(W) = \sum_{k=1}^{n} E[W_k W_m] - E[W_k]E[W_m]
\]

\[
= E[\sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k (1 + q_{\rho_k}(x))(1 + q_{A_k}(x))\sum_{m=1}^{n} \mu_{\rho_m} \mu_{A_m} l_m (1 + q_{\rho_m}(x))(1 + q_{A_m}(x))]
\]

\[
- \sum_{k=1}^{n} \mu_{\rho_k} \mu_{A_k} l_k \sum_{m=1}^{n} \mu_{\rho_m} \mu_{A_m} l_m
\]  
(4.6)

\[
= \sum_{k=1}^{n} \sum_{m=1}^{n} (E[q_{\rho_k}(x)q_{\rho_m}(x)] + E[q_{A_k}(x)q_{A_m}(x)]\mu_{\rho_k} \mu_{\rho_m} \mu_{A_k} \mu_{A_m} l_k l_m
\]

\[
= \sum_{k=1}^{n} \sum_{m=1}^{n} (\text{cov}(q_{\rho_k}(x),q_{\rho_m}(x)) + \text{cov}(q_{A_k}(x),q_{A_m}(x))\mu_{\rho_k} \mu_{\rho_m} \mu_{A_k} \mu_{A_m} l_k l_m
\]

Here, the expected value and covariance of the weight corresponding to the detailed structures are omitted.

4.1.2 Constraint formulation

For the sake of simplicity, only stress and displacement constraints are considered.

These constraints in the deterministic optimization can be formulated as,

Stress constraints:

\[
g_{\sigma_i} = \left| \frac{\sigma_i}{\sigma_{0i}} \right| - 1.0 \leq 0 \quad (i = 1, 2, ..., n_d)
\]  
(4.7)

Displacement constraints:

\[
g_{x_j} = \left| \frac{x_j}{x_{0j}} \right| - 1.0 \leq 0 \quad (j = 1, 2, ..., n_d)
\]  
(4.8)
where $\sigma_i$ is the design stress for the $i$-th element, $\sigma_{0i}$ is the permissible stress for the $i$-th element, $x_j$ is the $j$-th displacement component, $x_{0j}$ is the displacement limitation for the $j$-th displacement component. $n_s$ and $n_d$ are the numbers of stress and displacement constraints, respectively.

The stochastic behavior constraints $P(g_i \leq g_i^U) \geq p$ are modified to a form amenable to optimization calculation.

For stress constraint, the original stochastic behavior constraint may be written as

$$P(\left| \sigma_i \right| \leq \sigma_{i0}) \geq p \quad \text{or} \quad P(\left| \sigma_i \right| - \sigma_{i0} \leq 0) \geq p \quad (4.9)$$

where both $\sigma_i$ and $\sigma_{0i}$ are random variables. Assuming $\sigma_i$ is greater than zero, and denote $S$ as the difference between the two random variables ($\sigma_i$ and $\sigma_{0i}$), that is,

$$S = \sigma_i - \sigma_{0i} \quad (4.10)$$

Here, the new random variable $S$ is normalized to obtain a standard normal random variable, $\Phi$, with mean zero and variance 1,

$$\Phi = \frac{S - \mu_s}{\sigma_s} \quad (4.11)$$

Thus, Eq.(4.9) becomes

$$P(S \leq 0) \geq p \quad (4.12)$$

Then, minimizing $\mu_s$ from both sides of the inequality $S \leq 0$ and dividing $\sigma_s$ yield

$$P\left(\frac{S - \mu_s}{\sigma_s} \leq \frac{0 - \mu_s}{\sigma_s}\right) \geq p \quad \text{or} \quad P(\Phi \leq -\frac{\mu_s}{\sigma_s}) \geq p \quad (4.13)$$

This probability, $P(\Phi \leq x)$ is the definition of the cumulative distribution function for the standard normal, $F_\Phi$, Thus,
\[ F_\Phi(-\frac{\mu_s}{\sigma_s}) \geq p \]  

(4.14)

The minimum value of \( \Phi \) at which the probability level, \( p \) is satisfied is obtained from the inverse of the cumulative distribution function of the standard normal defined by \( \Phi^* \).

We have

\[ \Phi^*(p) = F^{-1}_\Phi(p) \quad \text{or} \quad \Phi^*(p) = -\frac{\mu_s}{\sigma_s} \]  

(4.15)

Since

\[ \mu_s = \mu_{\sigma_i} - \mu_{\sigma_{ai}} \]  

(4.16a)

\[ \sigma_s^2 = \sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2 \]  

(4.16b)

where \( \mu_{\sigma_i} \) and \( \mu_{\sigma_{ai}} \) are the mean values of the design stress and the permissible stress for the \( i \)-th element, respectively. \( \sigma_{\sigma_i} \) and \( \sigma_{\sigma_{ai}} \) are the standard derivation of the design stress and the permissible stress for the \( i \)-th element, respectively. Then

\[ \Phi^*(p) \leq -\frac{\mu_{\sigma_i} - \mu_{\sigma_{ai}}}{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2}} \]  

(4.17)

Expanding Eq.(4.17) yields

\[ \mu_{\sigma_i} - \mu_{\sigma_{ai}} + \Phi^*(p)\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2} \leq 0 \]  

(4.18)

Finally, normalizing with respect to the allowable mean of the stress for the \( i \)-th element gives

\[ \frac{\mu_{\sigma_i}}{\mu_{\sigma_{ai}}} - 1 + \Phi^*(p)\frac{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2}}{\mu_{\sigma_{ai}}} \leq 0 \]  

(4.19)

Similarly, the constraint expression for \( \sigma_i < 0 \) can be obtained as
\[
\frac{-\mu_{\sigma_i}}{\mu_{\sigma_{ai}}} - 1 + \Phi^*(p) \frac{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2}}{\mu_{\sigma_{ai}}} \leq 0
\]  
(4.20)

Therefore, the formulation for the stochastic stress constraints can be given by

\[
\left| \frac{\mu_{\sigma_i}}{\mu_{\sigma_{ai}}} \right| - 1 + \Phi^*(p) \frac{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{ai}}^2}}{\mu_{\sigma_{ai}}} \leq 0 \quad (i = 1, 2, \ldots, n_s) \]  
(4.21)

The formulation for the stochastic displacement constraints can be analogously obtained by

\[
\left| \frac{\mu_{x_j}}{\mu_{x_{aj}}} \right| - 1 + \Phi^*(p) \frac{\sqrt{\sigma_{x_j}^2 + \sigma_{x_{aj}}^2}}{\mu_{x_{aj}}} \leq 0 \quad (j = 1, 2, \ldots, n_d) \]  
(4.22)

It should be noticed that the stochastic behavior constraints have two parts. The first part is similar to a deterministic constraint specified on the mean values of the random variables. It corresponds to \( \Phi^*(p) = 0 \) i.e. \( p = 0.5 \). The second part contains the major contribution from the stochastic variables. \( \mu_{\sigma_i}, \mu_{x_j}, \sigma_{\sigma_i}, \sigma_{x_j} \) are the mean values and the standard derivation, respectively, of the response variables that are obtained from stochastic calculations.

It should be emphasized that in the above derivation an assumption that \( \mu_{\sigma_i} \gg \sigma_{\sigma_i} \) and \( \mu_{x_j} \gg \sigma_{x_j} \) is made. This is reasonable when the constraint is active.

For the constraints with the prescribed upper and lower bounds, on the other hand, the mean values of the design variables are utilized. For the equality constraints, the stochastic behavior constraints including the deterministic and stochastic components may be obtained through the perturbation technique. However, CometBoards’ interface provides for inequality constraints only at present.
4.2 Formulation of design optimization and SQP algorithm in CometBoards

The optimum design problems for a minimum weight of random structural system subject to maximum stress and displacement constraints can be generally formulated as

Minimize \( W = \overline{W}(\overline{b}, \overline{x}) \) \hspace{1cm} (4.23)

Subject to \( P(g_i(b, x) \leq 0) \geq p \hspace{1cm} (i=1,2,\ldots,m_s + m_d) \) \hspace{1cm} (4.24a)

\[ b_L \leq \overline{b} \leq b_U \] \hspace{1cm} (4.24b)

where \( \overline{W} \) is taken as the expected merit function, \( b \) and \( x \) represent the vector of design variables and random variables respectively. \( \overline{b} \) and \( \overline{x} \) are the above vectors evaluated at the mean values. \( p \) is the satisfaction probability of the \( i \)-th constraint. \( b_L \) and \( b_U \) represent a vector of lower and upper values of design variables, respectively.

In accordance with section 4.1, the formulation of the design problem can be described in detail as

Minimize \( W = \sum_{i=1}^{n} \mu_{\rho_i} \mu_{A_i} l_i \) \hspace{1cm} (4.25)

Subject to \( g_{\sigma_i}(A) = \left| \frac{\mu_{\sigma_i}}{\mu_{\sigma_{0i}}} \right| -1 + \phi^*(p) \frac{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_{0i}}^2}}{\mu_{\sigma_{0i}}} \leq 0 \hspace{1cm} i=1,\ldots,n \) \hspace{1cm} (4.26a)

\( g_{X_j}(A) = \left| \frac{\mu_{X_j}}{\mu_{X_{0j}}} \right| -1 + \phi^*(p) \frac{\sqrt{\sigma_{X_j}^2 + \sigma_{X_{0j}}^2}}{\mu_{X_{0j}}} \leq 0 \hspace{1cm} j=1,\ldots,m \) \hspace{1cm} (4.26b)

\( A_i^L \leq A_i \leq A_i^U \) \hspace{1cm} (4.26c)

where \( A_i \) is the cross-sectional area of the \( i \)-th element and design variables. \( A_i^L \) and \( A_i^U \) are the lower and upper bounds of the \( i \)-th design variable.
It is clearly evident that the responses in Eq.(4.26a) and Eq.(4.26b) can be evaluated through the stochastic calculations. The formulation for the stochastic optimization is intrinsically a highly nonlinear program. To cope with the complicated optimization problems, the sequential quadratic programming technique (SQP) in CometBoards has been used in this study.

CometBoards can evaluate the performance of different optimization algorithms and analysis methods while solving a problem. CometBoards has three different analysis methods and one dozen optimization algorithms. It has a modular organization with a soft coupling feature, which allows quick integration of new or user-supplied analyzers and optimizers without any change to the source code. The CometBoards code reads information from data files; formulates design as a sequence of subproblems; and generates the optimum solution. CometBoards can be used to solve a large problem, definable through multiple disciplines. It can improve an existing system by optimizing a small portion of a large problem. It includes all formulations, strategy, technique, approximate methods and so forth. These features assist convergence and reduce the amount of CPU time required to solve difficult optimization problems of the aerospace industry. CometBoards has been successfully used to solve the structural design of the international space station components, the design of the nozzle components of an air-breathing engine, and airframe and engine synthesis for subsonic and supersonic aircraft and so on. The modular organization of CometBoards is depicted in Fig. 4.1.

Here, the probabilistic modules and related modules are briefly introduced. CometBoards is adding the probabilistic design function through the two modules: “Probabilistic optimization” and “Stochastic analysis”. Stochastic analysis employs both
the primal and dual integrated force method for mechanical, thermal and setting of support loads by using the perturbation technique. The probabilistic module formulates design as a stochastic optimization problem and generates its solution. Structural design refers to design of structures through regular optimization or a subproblem strategy. In the problem formulation and solution module, information is read from data files, the design is cast as a sequence of optimization subproblems and the solution is obtained. CometBoards accommodates the Fortran 77 language and is available in the Unix operating system on workstations.

In CometBoards, there are three optimization codes based on the sequential quadratic programming technique. The first on is the DNCONG [93] routine of the IMSL library. It solves the nonlinear problem as a sequence of quadratic subproblems. The subproblem is obtained by performing a quadratic approximation of the objective
function and liberalizing the constraints. At the \( k \)-th intermediate design the following quadratic problem is solved

\[
\text{Minimize} \quad \frac{1}{2} d^T B_k d + \nabla f(x)^T d \quad d \in \mathbb{R}^n
\]  

(4.27)

\[
\text{Subject to} \quad \nabla g_i(x_k)^T d + g_i(x_k) \geq 0 \quad (i = 1, 2, \ldots, m)
\]  

(4.28a)

\[
x^L - x_k \leq d \leq x^U + x_k
\]  

(4.28b)

where \( d \) is the solution of subproblem. \( B_k \) is the positive definite approximation to the Hessian of objective function \( f \), \( x_k \) is design vector at \( k \)-th iteration. \( \nabla g_i(x_k) \) is the gradient vector evaluated at \( x_k \).

The step length \( \alpha_k \) is calculated by suing a simple Armijo type of bisection method combined with quadratic interpolation to decrease the augmented Lagrangian function.

The second is the NPSOL in NAG [94], which also uses the augmented Lagrangian. The search direction is generated through a quadratic subproblem while step length is calculated using an augmented Lagrangian, which is designed to avoid discontinuities as much as possible.

The last one is the IDESIGN [95] sequential quadratic programming method based on Pshenichny’s recursive quadratic programming method. The matrix \( [B_k] \) approximates the Lagrangian function

\[
L(x, \lambda) = f(x) + \sum \lambda_j g_j(x) \quad j \in I_A
\]  

(4.29)
where $\lambda_j$ are the Lagrangian multipliers associated with the active constrains. The step length $\alpha_k$ is obtained by minimizing the composite decent functions of the objective and the constraints.

Each of the sequential quadratic programming methods is briefly described here. Refer to specific references for further details on these methods. It is shown in Chapter 5 that the SQP can efficiently and effectively solve the stochastic optimization problems with not only the expected objective function, but also the robust design.

4.3 Formulation of robust design optimization

For the design optimization of structures with random variables, as previously stated, the conventional method is only to minimize the expected value of the objective function. In fact, the design utilized the reliability-based method to analyze the probability of failure of a system. It only concerns about the rare events based on the probability of failure occurrence shown in Fig.4.2.

![Fig.4.2 The difference between robustness and reliability in design optimization.](image)

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The structural performance, however, defined by objectives and constraints, is also varied with the variation on design variables at different period of the service, even causes the constraint conditions invalid, make design unfeasible. To improve the design, the deviation of structure performance caused by the fluctuation of random variables should also be minimized, without eliminating these factors. Therefore, the robust design is introduced. The objective of robust design is to optimize the mean performance and minimize its variation, while maintaining feasibility with probabilistic constraints. The feasibility of the design can make the performance less sensitive to the scatter of the parameters. Hence, robust design concerns about the probability distribution around the mean values.

Before discussing the formulation of robust design, assume the formulation of the deterministic optimization problem is given as

Minimize \( f(x) \) \hspace{1cm} (4.30)

Subject to \( g_i(x) \leq 0 \) \hspace{1cm} (i = 1, 2, \ldots, k) \hspace{1cm} (4.31a)

\[ x_L \leq x \leq x_U \] \hspace{1cm} (4.31b)

where \( x, x_L \) and \( x_U \) are vectors for design variables, lower bounds and upper bounds, respectively. The objective function is denoted by \( f(x) \). \( g_i(x) \) denotes the \( i \)-th constraint function.

Because of the stochastic parameters, the mean value and the standard deviation of the objective function can be presented statistically as

\[ \mu_f = E[f(x)] = \int \ldots \int f(x) p(x_1, x_2, \ldots, x_k) dx_1 dx_2 \ldots dx_k \] \hspace{1cm} (4.32a)
\[ \sigma_f = E[(f(x) - \mu_f)^2] \]
\[ = \int \ldots \int (f(x) - \mu_f)^2 p(x_1, x_2, \ldots, x_k) dx_1 dx_2 \ldots dx_k \]  
\[ (4.32b) \]

where \( p(x_1, x_2, \ldots, x_k) \) is the joint probability density function.

In most practical applications, it is impossible to know the joint probability density functions. Although it is often assumed that all variables have independent normal distributions, evaluating Eq.(4.32a) and Eq.(4.32b) are extremely time consuming and computationally expansive. Hence, it is reasonable to introduce an approximation method using SFEM.

Based on SFEM and the multiple criteria of robust design, the robust design optimization problem can be formulated by

Minimize \[ \tilde{f} = [\mu_f, \sigma_f] \]  
\[ (4.33) \]

Subject to \[ E[g_i(x)] + \beta_i \sigma(g_i(x)) \leq 0 \quad (i = 1, 2, \ldots, k) \]  
\[ (4.34a) \]
\[ x_L \leq x \leq x_v \]  
\[ (4.34b) \]

where \( \tilde{f} \) is a vector of objectives, which instanenously minimizes the mean value \( \mu_f \) and the standard deviation \( \sigma_f \) of the objective. \( \beta_i \) is a constant that reflects the probability satisfied by the \( i \)-th constraint.

In the robust design optimization problem, optimization of the mean value often conflicts with the minimization of the variance in the objective function. Therefore, a trade-off decision between them must be made. Pareto optimality is one way of determining the multi-objective [96]. However, this method may be impossible to achieve some optimal solutions and there is no guarantee that the best design can be
selected, while the feasible region in the objective function space is not convex. But such a case is seldom encountered in practice.

In the Pareto optimum, the vector of objective functions is usually tackled as the scalarized objective function. Although there are several scalarization approaches [85] such as the Lexieographic method and the goal programming, the linear combination method, which makes the individual objective nearly combine by the weighting factor as a scalar objective function, is usually adopted because of its simplicity and ease operation. Therefore, the formulation of the robust design problem can be expressed by

\[
\begin{align*}
\text{Minimize} & \quad \tilde{f} = \frac{(1-\alpha)f^*}{\mu_f} + \alpha \frac{\sigma_f}{\sigma_f} \\
\text{Subject to} & \quad E[g_i(x)] + \beta_i \sigma(g_i(x)) \leq 0 \quad i = 1, 2, \ldots, k \\
& \quad x_L \leq x \leq x_U
\end{align*}
\]

where \(\alpha\) is the weighting factor, \(0 < \alpha < 1\), which is determined depending on the importance and robustness of each objective function. \(\mu_f^*\) and \(\sigma_f^*\) are normalization factors, which are the function values at the optimum considering only the mean and the standard deviation, respectively.

Setting

\[
\mu_w = E[W] = \sum_{k=1}^{n} \mu_{\rho_k} \mu_{\lambda_k} l_k
\]

\[
\sigma_w = \sqrt{\text{cov}(W)} = \sqrt{\sum_{k=1}^{n} \sum_{m=1}^{n} \left( \text{cov}(q_{\rho_k} q_{\rho_m}) + \text{cov}(q_{\lambda_k} q_{\lambda_m}) \right) \mu_{\rho_k} \mu_{\rho_m} \mu_{\lambda_k} \mu_{\lambda_m} l_k l_m}
\]

Then, the formulation of robust design problem corresponding to the optimal mean value and variance of the weight of structures, can be expressed by
Minimize \[ f_W = (1 - \alpha) \frac{\mu_w}{\mu_w} + \alpha \frac{\sigma_w}{\sigma_w} \] (4.38)

Subject to \[ g_{\sigma_i}(A) = \left| \frac{\mu_{\sigma_i}}{\mu_{\sigma_i}} \right| - 1 + \phi^*(p) \frac{\sqrt{\sigma_{\sigma_i}^2 + \sigma_{\sigma_i}^2}}{\mu_{\sigma_i}} \leq 0 \quad i = 1, 2, \ldots, n \] (4.39a)

\[ g_{\sigma_j}(A) = \left| \frac{\mu_{\sigma_j}}{\mu_{\sigma_j}} \right| - 1 + \phi^*(p) \frac{\sqrt{\sigma_{\sigma_j}^2 + \sigma_{\sigma_j}^2}}{\mu_{\sigma_j}} \leq 0 \quad j = 1, 2, \ldots, m \] (4.39b)

\[ A_i^L \leq A_i \leq A_i^U \] (4.39c)

where \( \mu^*_w \) and \( \sigma^*_w \) are the optimal value of the objective function considering only the mean and the standard deviation, respectively.

It should be noted that the design problem of Eq.(4.38) is converted to the optimization problem for a pure mean value of weight while \( \alpha = 0 \) or for a pure standard deviation of weight while \( \alpha = 1 \).

In the present study, the robust design optimization problem, expressed by Eq.(4.38) and (4.39), can be efficiently solved using the sequential quadratic programming method in CometBoards.
Historically, Monte Carlo simulation was considered to be a technique, using random or pseudorandom numbers, for solution of a model. In the last two decades, much effort has been done to develop reliable and efficient methods for probabilistic analysis of engineering structures. Although several of the well-known approaches have been developed, such as the perturbation method, Neumann expansion method with Monte Carlo simulation, linear partial derivation method and the like, Monte Carlo simulation technique is still of considerable usefulness for solving stochastic problems in engineering practice. It is usually an accurate, simple and versatile approach. However, this technique is computationally intensive and time consuming due to its accuracy depending on the sample size.

In this chapter, the modal decomposition method, the direct Monte Carlo simulation and Latin Hypercube sampling are described. Then, 15 numerical examples are provided to illustrate the produces of the stochastic analysis, stochastic sensitivity analysis and stochastic optimization. Some analytical results are compared with the Neumann expansion, Monte Carlo simulation and ANSYS.
5.1 Monte Carlo simulation

In the classical Monte Carlo simulation, a large number of realizations of the random variables are generated based on their specific statistical description, a result is calculated for each realization as in deterministic analysis, and the results are examined by statistical methods. It is obvious that the method is completely general for linear or nonlinear analysis, and has been used to calibrate and validate all other techniques. Since this method is expansive in terms of computing time, some improvements using variance reduction technique are introduced, like importance sampling and Latin Hypercube sampling.

5.1.1 Discretization of the stochastic field

In order to perform the stochastic analysis using Monte Carlo simulation[25], the random fields must be generated, the discretized Gaussian homogeneous stochastic fields can be simulated and their sample functions can be generated with the aid of the covariance matrix decomposition, called Cholesky decomposition. This method makes it possible to transform a set of independent Gaussian random variables into a set of correlated Gaussian random variables with a prescribed covariance matrix.

For the purpose, the structure is divided into an appropriate number of small finite elements in such way that the property values within each element can be considered approximately constant. If there are n elements in the total structures, then n property values associate with these n elements. Thus, assuming that \( q(x) \) is a homogeneous Gaussian process with zero mean value, then the values of \( q_i = q(x_i) \), \( i = 1,2,\ldots,n \), are random with mean zero, but correlated, where \( x_i \) indicate the element centroid.
The covariance matrix of $q(x)$ can be specified by

$$
C_{qq} = E[q_i q_j] = \text{cov}(q_i, q_j) = \begin{bmatrix}
\text{Var}(q_1) & \text{Cov}(q_1, q_2) & \ldots & \text{Cov}(q_1, q_n) \\
\text{Cov}(q_1, q_2) & \text{Var}(q_2) & \ldots & \text{Cov}(q_2, q_n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(q_1, q_n) & \text{Cov}(q_2, q_n) & \ldots & \text{Var}(q_n)
\end{bmatrix}
$$

(5.1)

The correlated random vector $\{q\} = \{q_1, q_2, q_3, \ldots, q_n\}^T$ can be obtained by

$$
\{q\} = [L][Z]
$$

(5.2)

where $\{Z\}$ is a vector consisting of $n$ independent Gaussian random variables with zero mean and unit standard deviation. $[L]$ is a lower triangular matrix obtained by the Cholesky decomposition of the covariance matrix $C_{qq}$.

Since $E[\{q\}] = 0$ and $C_{qq} = [L][L]^T$, the expected value of $Z$ is then given by

$$
E[\{Z\}] = E[[L]^{-1}\{q\}] = [L]^{-1}E[\{q\}] = [0]
$$

(5.3)

and the covariance matrix of $\{Z\}$ can be shown to be

$$
= E[[L]^{-1}\{q\} \{q\}^T[L]^{-T}] \\
= [L]^{-1}E[\{q\} \{q\}^T] [L]^{-T} \\
= [L]^{-1}C_{qq} [L]^{-T} \\
= [L]^{-1}[L][L]^T[L]^{-T} \\
= [I]
$$

(5.4)

Once the Cholesky decomposition is accomplished, different samples of $\{q\}$ are easily given from Eq.(5.2).

It should be noted that to approach the original covariance matrix, the simulated covariance matrix needs a fairly large number of sample size.
5.1.2 Direct Monte Carlo simulation method

The direct Monte Carlo simulation method [97] is also called the rude Monte Carlo simulation method. It is based on randomly sampling the values of the random input variables for each execution run.

In direct Monte Carlo simulation, N samples of the vector of random variables are generated from known or assumed joint probability density function. The implementation of the method consists of these samples in numerical simulation. The procedure used in the deterministic analysis is repeated for each sample of the simulation process. N responses are then obtained to compute the statistical moments of the response.

Firstly, random sampling involves repeatedly forming random vectors of parameter from prescribed probability distributions. For example, a normally-distributed random variable \( x \) with mean \( \mu \) and standard deviation \( \sigma \) can be generated by

\[
x^* = r_n \sigma + \mu
\]  

(5.5)

where \( r_n \) is normally distributed random numbers with mean 0 and variance 1.

For a multivariate normal distribution with covariance matrix \( C_{qq} \), the matrix \( C_{qq} \) must be first decomposed by Cholesky factorization as

\[
C_{qq} = [L][L]^T
\]  

(5.6)

Then, the random variables vector \( \{X\} \) can be written as

\[
\{X\} = \{L\}\{r_n\} + \{\mu\}
\]  

(5.7)

where \( \{L\} \) is the lower triangular matrix. \( \{r_n\} \) is the independent normal random numbers with mean 0 and variance 1.
The procedure is repeated for sample size $N$, resulting in a set of variables with expected mean vector $\{\mu\}$ and covariance matrix $C_{qq}$.

Therefore, after defining the stochastic field and sampling, the direct Monte Carlo simulation performs the analyses of all simulated samples using the deterministic finite element method. For the statistical responses, such as internal forces, displacements stresses and strains, the expected value of the quantities $Q$ of each response can be written as

$$Q_0 = E[Q] = \frac{\sum_{i=1}^{N} Q_i}{N}$$

and the covariance matrix is given by

$$\text{Cov}(Q, Q^T) \approx \frac{\sum_{i=1}^{N} (Q_i - Q_0)(Q_i - Q_0)^T}{N}$$

where $Q_i$ represents the $i$-th sample vector of response.

5.1.3 Latin Hypercube sampling method

Although Monte Carlo simulation is a powerful tool to solve random problems, this sampling scheme in direct Monte Carlo simulation approach has a serious drawback, which requires many samples for good accuracy and repeatability.

To reduce the required number of samples, one of the best small-sample Monte Carlo approaches is Latin hypercube sampling (LHS). Latin hypercube sampling constructs a highly dependent joint probability density function for the random variables in the problem, which allows good accuracy in the response parameters using only a
small number of samples. It was first proposed by Mckay et. al. [98] and has been further
developed for different purposes by several researches, e.g. [99-104].

Let \( N \) and \( K \) denote the required number of samples and the number of random
variables. The sampling space is then \( K \)-dimensional. A \( N \times K \) matrix \([P]\), in which each
of the \( K \) columns is a random permutation of \( 1,...,N \) and a \( N \times K \) matrix \([R]\) of
independent random numbers from the uniform \((0,1)\) distribution are established. These
matrices form the basis sampling plan, represented by the matrix \( T \) as

\[
[T] = \frac{([P] - [R])}{N} \tag{5.10}
\]

According to the target marginal distribution, each element of \([T]\), \( t_{ij} \) is then mapped as

\[
\hat{x}_{ij} = F_{x_j}^{-1}(t_{ij}) \tag{5.11}
\]

where \( F_{x_j}^{-1} \) is the inverse of the target cumulative distribution function for variable \( j \).

Thus, each row, \( \hat{X}_i = [\hat{x}_{i1}, \hat{x}_{i2}, ..., \hat{x}_{ik}] \) in \( [X] \) now contains input data for one deterministic
computation. Fig. 5.1 illustrates an LHS sample in a two-dimensional space with sample
size \( N = 5 \).

\[
\begin{bmatrix}
2 & 4 \\
3 & 2 \\
5 & 3 \\
1 & 5 \\
4 & 1 \\
\end{bmatrix} \begin{bmatrix}
0.80 & 0.63 \\
0.74 & 0.40 \\
0.18 & 0.91 \\
0.28 & 0.45 \\
0.77 & 0.08 \\
\end{bmatrix} \begin{bmatrix}
0.24 & 0.67 \\
0.45 & 0.32 \\
0.14 & 0.91 \\
0.65 & 0.18 \\
\end{bmatrix}
\]

Fig. 5.1 A Latin Hypercube sample with \( N=5 \) for two independent variables, \( x_1 \) and \( x_2 \).

The method, however, is based on the assumption that the variables are
independent of each other. In reality, most of the input variables are correlated to some
extent. Random pairing of correlated variables may result in impossible combinations and bias the uncertainty.

Iman and Conover [105] proposed a method to induce correlation among the variables. The variables are paired based on the rank correlation of some target values in this method. It has been shown that such a spurious correlation can be reduced by modifications in the permutation matrix $[P]$, the elements of $[P]$, $p_{ij}$ are divided by the number of samples plus one, and mapped on the Gaussian distribution with mean zero and standard deviation one as

$$y_{ij} = \Phi^{-1}_{(0,1)} \left( \frac{p_{ij}}{N + 1} \right)$$

(5.12)

Then, the covariance matrix of $[Y]$ is evaluated and by using the Cholesky decomposition, we have

$$Cov([Y]) = [L][L]^T$$

(5.13)

where $[L]$ is the lower triangular matrix. A new matrix $[Y^*]$ with a sample covariance equal to the identity is computed as

$$[Y^*] = [Y][L]^{-T}$$

(5.14)

and the ranks of the elements of the columns of $[Y^*]$ become the elements in the columns of the matrix $[P^*]$. If the elements of $[P]$ in Eq.(5.10) are replaced by the elements of this matrix, the sampling matrix $[S]$ will contain a considerably lower amount of undesired correlation. However, the Cholesky decomposition of Eq.(5.13) requires that $Cov([Y])$ is positive definite, which means the number of samples is higher than the number of stochastic variables, i.e, $N > K$. 

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If the target correlation matrix is different from unity, then the target correlation is applied by replacing Eq.(5.14) with

\[
[Y^*] = [Y][L]^{-T}[L]^T
\]  

(5.15)

where \([L]\) is the lower triangular matrix from the Cholesky decomposition of the target correlation matrix.

The correlation of the sample \([\hat{x}]\) in Eq.(5.11) will approach the target correlation exactly if the stochastic variables are Gaussian, and approximately if the stochastic variables are non-Gaussian.

5.2 Illustrative examples

The stochastic response analysis, stochastic response sensitivity analysis and stochastic design optimization are illustrated in fifteen examples, which include a number of indeterminate problems, some with thermal loads and support settling. In these examples, the sizing design variables, material properties, and loads are treated as random variables. The load types include mechanical, thermal and settling of support and their combinations. The geometrical parameters of each structure, like length of a beam or bar truss, dimensions of a membrane, are considered deterministic variables.

The stochastic responses are obtained by using Maple V and FORTRAN codes (see Appendix 1). The stochastic response sensitivities are calculated by using Maples V. The stochastic optimum solutions are solved by using CometBoards. These closed form solutions are verified through Monte Carlo simulation using Maple V.

The stochastic response calculations are obtained in the following steps:

1. Definition of the primitive random variables
A set of stochastic parameters, i.e. sizing design variables, material properties and loads, are identified as primitive random variables. These variables are specified through their means and covariance. Each primitive random variable is defined by a normalized random variable, \( q \).

2. Deterministic analysis

The deterministic solutions are obtained through the mean values of random variables.

3. Definition of stochastic matrices

The stochastic matrices and their derivatives with respect to the normalized random variable, \( q \), are expressed in terms of the normalized primitive random variables.

4. Stochastic response calculation

The mean and covariance of responses, such as force, displacement and stress are calculated using the stochastic formulas derived from IFM and IFMD given in earlier sections.

5. Description of stochastic responses

The probability density function and cumulative distribution function are illustrated for selected response parameters.

Similarly, the stochastic sensitivity analysis with respect to the primitive random variables has the same steps for evaluation after the deterministic sensitivity formulas.

To obtain the stochastic optimal solution, the following steps are recommended:

1. The user-supplied analyzers for stochastic analysis are programmed by Fortran code.
2. The data files including *.input, *.imod, *.outmodes and *.dsngdat need to be prepared for each problem.
3. Using the Gen-Sim version of CometBoards, the stochastic optimal problems are run and create the optimal results.

5.2.1 15 numerical examples of stochastic, sensitivity analysis and optimization

Example 1: Theromomechanical solution for a fixed column.

A column of length $3l$ ($l = 10\text{in.}$) shown Fig. 5.2(a), is restrained at both ends. It is made of steel with a Young’s modulus $E$ of mean, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{E_1} \\
\mu_{E_2} \\
\mu_{E_3}
\end{bmatrix} = \begin{bmatrix} 30,000 \\
30,000 \\
30,000 \end{bmatrix} \text{ksi} \quad \begin{bmatrix}
\sigma_{E_1} \\
\sigma_{E_2} \\
\sigma_{E_3}
\end{bmatrix} = \begin{bmatrix} 822.0 \\
822.0 \\
822.0 \end{bmatrix} \text{ksi} \quad [\rho_E] = \begin{bmatrix} 1.000 & 0.665 & 0.600 \\
0.665 & 1.000 & 0.600 \\
0.600 & 0.600 & 1.000 \end{bmatrix}.
$$

The coefficients of thermal expansion, $\alpha$, have mean values, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{\alpha_1} \\
\mu_{\alpha_2} \\
\mu_{\alpha_3}
\end{bmatrix} = \begin{bmatrix} 6.0 \\
6.0 \\
6.0 \end{bmatrix} \times 10^{-6} \text{°F} \quad \begin{bmatrix}
\sigma_{\alpha_1} \\
\sigma_{\alpha_2} \\
\sigma_{\alpha_3}
\end{bmatrix} = \begin{bmatrix} 3.0 \\
3.0 \\
3.0 \end{bmatrix} \times 10^{-7} \text{°F} \quad [\rho_{\alpha}] = \begin{bmatrix} 1.000 & 0.800 & 0.600 \\
0.800 & 1.000 & 0.600 \\
0.600 & 0.600 & 1.000 \end{bmatrix}.
$$

A uniform temperature variation, $T$, along the entire length of the column has a mean, $\mu_T$, of $10.0 \text{°F}$ and a standard deviation, $\sigma_T$ of $0.447 \text{°F}$. The sizing variables, column areas, have the mean, standard deviation and correlation coefficient matrix as follows,

$$
\begin{bmatrix}
\mu_{A_1} \\
\mu_{A_2} \\
\mu_{A_3}
\end{bmatrix} = \begin{bmatrix} 1.0 \\
2.0 \\
1.0 \end{bmatrix} \text{in.}^2 \quad \begin{bmatrix}
\sigma_{A_1} \\
\sigma_{A_2} \\
\sigma_{A_3}
\end{bmatrix} = \begin{bmatrix} 0.0894 \\
0.179 \\
0.0775 \end{bmatrix} \text{in.}^2 \quad [\rho_A] = \begin{bmatrix} 1.000 & 0.626 & 0.361 \\
0.625 & 1.000 & 0.360 \\
0.361 & 0.360 & 1.000 \end{bmatrix}.
$$

It is also subjected to two mechanical loads ($P_1$ and $P_2$) applied at the one-third and two-third span locations, respectively. The loads have the mean, standard deviation and correlation coefficient matrix as follows:
\[
\begin{align*}
\begin{bmatrix}
\mu_{P_1} \\
\mu_{P_2}
\end{bmatrix} &= \begin{bmatrix} 10.0 \\ 20.0 \end{bmatrix} \text{kip} \\
\begin{bmatrix}
\sigma_{P_1} \\
\sigma_{P_2}
\end{bmatrix} &= \begin{bmatrix} 0.548 \\ 1.342 \end{bmatrix} \text{kip} \\
\rho_P &= \begin{bmatrix} 1.000 & 0.680 \\
0.680 & 1.000 \end{bmatrix}.
\end{align*}
\]

(a) Column under mechanical and thermal load. (b) Analysis model.

Fig. 5.2 Thermomechanical analysis of a three-span column.

Therefore, we note that the problem has twelve stochastic variables, which in terms of their normalized primitive random variables are:

\[
\begin{align*}
A_1 &= \mu_{A_1} \left(1 + q_{A_1}\right) = \mu_1 \left(1 + q_1\right) \\
A_2 &= \mu_{A_2} \left(1 + q_{A_2}\right) = \mu_2 \left(1 + q_2\right) \\
A_3 &= \mu_{A_3} \left(1 + q_{A_3}\right) = \mu_3 \left(1 + q_3\right) \\
E_1 &= \mu_{E_1} \left(1 + q_{E_1}\right) = \mu_4 \left(1 + q_4\right) \\
E_2 &= \mu_{E_2} \left(1 + q_{E_2}\right) = \mu_5 \left(1 + q_5\right) \\
E_3 &= \mu_{E_3} \left(1 + q_{E_3}\right) = \mu_6 \left(1 + q_6\right) \\
\alpha_1 &= \mu_{\alpha_1} \left(1 + q_{\alpha_1}\right) = \mu_7 \left(1 + q_7\right) \\
\alpha_2 &= \mu_{\alpha_2} \left(1 + q_{\alpha_2}\right) = \mu_8 \left(1 + q_8\right) \\
\alpha_3 &= \mu_{\alpha_3} \left(1 + q_{\alpha_3}\right) = \mu_9 \left(1 + q_9\right) \\
P_1 &= \mu_{P_1} \left(1 + q_{P_1}\right) = \mu_{10} \left(1 + q_{10}\right) \\
P_2 &= \mu_{P_2} \left(1 + q_{P_2}\right) = \mu_{11} \left(1 + q_{11}\right) \\
T &= \mu_T \left(1 + q_T\right) = \mu_{12} \left(1 + q_{12}\right)
\end{align*}
\]
The normalized primitive random variable, \( \{ q \} \), with zero mean and standard deviation given by the ratio of the standard deviation to the mean of the corresponding stochastic variable is assumed to be of order \( O(1) \). This justifies the use of the Taylor series expansion in \( \{ q \} \).

The stochastic response can be calculated by using the previous formulas of stochastic analysis. The solution to the three columns follows,

**Force:**

\[
\{ \mu_F \} = \{ \mu_F \}^I = \begin{bmatrix} 11.840 \\ 1.840 \\ -18.160 \end{bmatrix} \text{kip} \quad \{ \mu_F \}^II = \begin{bmatrix} 11.841 \\ 1.841 \\ -18.159 \end{bmatrix} \text{kip}
\]

\[
\{ \sigma_F \} = \begin{bmatrix} 1.004 \\ 0.741 \\ 1.144 \end{bmatrix} \text{kip} \quad [\rho_F] = \begin{bmatrix} 1.000 & 0.845 & -0.341 \\ 0.845 & 1.0 & 0.034 \\ -0.341 & 0.034 & 1.0 \end{bmatrix}
\]

**Displacement:**

\[
\{ \mu_x \} = \{ \mu_x \}^I = \begin{bmatrix} -0.455 \\ -0.545 \end{bmatrix} \times 10^{-2} \text{in.} \quad \{ \mu_x \}^II = \begin{bmatrix} -0.457 \\ -0.548 \end{bmatrix} \times 10^{-2} \text{in.}
\]

\[
\{ \sigma_x \} = \begin{bmatrix} 0.434 \\ 0.504 \end{bmatrix} \times 10^{-3} \text{in.} \quad [\rho_x] = \begin{bmatrix} 1.0 & 0.977 \\ 0.977 & 1.0 \end{bmatrix}
\]

**Stress:**

\[
\{ \mu_\sigma \} = \{ \mu_\sigma \}^I = \begin{bmatrix} 11.840 \\ 0.920 \\ -18.160 \end{bmatrix} \text{ksi} \quad \{ \mu_\sigma \}^II = \begin{bmatrix} 11.914 \\ 0.924 \\ -18.237 \end{bmatrix} \text{ksi}
\]

\[
\{ \sigma_\sigma \} = \begin{bmatrix} 1.272 \\ 0.369 \\ 1.472 \end{bmatrix} \text{ksi} \quad [\rho_\sigma] = \begin{bmatrix} 1.000 & 0.497 & -0.931 \\ 0.497 & 1.000 & -0.580 \\ -0.931 & -0.580 & 1.000 \end{bmatrix}
\]

where the elements of \([\rho_F]\), \([\rho_x]\) and \([\rho_\sigma]\) are the correlation coefficients. It can be noted that there is a very little difference between the second-order approximate mean values and deterministic solutions or the first-order approximate mean values. However, the standard deviations of the column forces are about 8% for the first column, 40% for the second column and 6% for the third column; the standard deviations of the displacements
are 9.5% and 9.2%; the standard deviations of the stresses are about 11% for the first column, 40% for the second column and 8.1% for the second column.

The probability density function and the cumulative distribution function for the displacements are depicted in Fig. 5.3(a) and (b), respectively. There is a little difference between the first- and second-order approximations in the displacement probabilistic description.

![Probability density function for two displacements, $x_1, x_2$.](image)

![Cumulative distribution functions for two displacements, $x_1, x_2$.](image)

Fig. 5.3 Stochastic description of displacements in the fixed column.
The values of the response variables for different probability levels of occurrence 
\((p = 50\%, 25\% \text{ and } 75\%)\) are listed in Table 5.1. In the table, about four to five percent change is found in the third and first column forces in the \(p = 25\% \text{ and } 75\%\) levels, about 27\% change is noted in the second column force; about 6\% changes are found in two sectional displacements. Whereas about five to seven percent change is found in the third and first column stresses, about 27\% change is noted in the second column stress.

Table 5.1 Response values for \(p\)-percent probability of success in the fixed column.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>(p = 50%)</th>
<th>(p = 25%)</th>
<th>(p = 75%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column Force</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(F_1) (kip)</td>
<td>11.840</td>
<td>11.163 (94.28%)</td>
<td>12.517 (105.72%)</td>
<td></td>
</tr>
<tr>
<td>(F_2) (kip)</td>
<td>1.840</td>
<td>1.340 (72.83%)</td>
<td>2.340 (127.17%)</td>
<td></td>
</tr>
<tr>
<td>(F_3) (kip)</td>
<td>-18.160</td>
<td>-17.388 (95.75%)</td>
<td>-18.932 (104.25%)</td>
<td></td>
</tr>
<tr>
<td>Displacements</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_1) (in.)</td>
<td>-0.00457</td>
<td>-0.00428 (93.65%)</td>
<td>-0.00487 (106.56%)</td>
<td></td>
</tr>
<tr>
<td>(x_2) (in.)</td>
<td>-0.00548</td>
<td>-0.00514 (93.80%)</td>
<td>-0.00582 (106.20%)</td>
<td></td>
</tr>
<tr>
<td>Column Stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\sigma_1) (ksi)</td>
<td>11.914</td>
<td>11.056 (92.80%)</td>
<td>12.773 (107.21%)</td>
<td></td>
</tr>
<tr>
<td>(\sigma_2) (ksi)</td>
<td>0.924</td>
<td>0.675 (73.05%)</td>
<td>1.173 (126.95%)</td>
<td></td>
</tr>
<tr>
<td>(\sigma_3) (ksi)</td>
<td>-18.237</td>
<td>-17.244 (94.56%)</td>
<td>-19.230 (105.45%)</td>
<td></td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity at the 75\% probability of occurrence level, with respect to the primitive random variables, is shown in Fig. 5.4(a), (b), and (c) for the first column force, the displacement, \(x_1\), and stress in the first column, respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the force sensitivity, the first column force is sensitive to the loads \(P_2\) and \(P_1\), column areas \(A_1\) and \(A_3\), Young’s modulus \(E_1\) and \(E_3\), not sensitive to the second column area \(A_2\) and its Young’s modulus \(E_2\). In the displacement sensitivity, the sensitivity of the displacement \(x_1\) is similar to the
(a) Sensitivity for the first column force.

(b) Sensitivity for the displacement $x_1$.

(c) Sensitivity for stress in the third column.

Fig. 5.4 Sensitivity analysis of responses in the fixed column.
sensitivity of the first column force. In the stress sensitivity, the third column stress is sensitive to the load $P_2$, column area $A_3$ and its Young’s modulus $E_3$, not sensitive to the column area $A_2$ and its Young’s modulus $E_2$.

In stochastic optimization, the material density is assumed to have a mean of 0.289 lb/in.$^3$ and standard deviation of 0.005 lb/in.$^3$. The allowable mean value of strength and standard deviation are 10,000 psi and 1000 psi, respectively. The allowable displacement has a mean of 0.003 in. and standard deviation of 0.0003 in. $0.00\text{in.}^2 \leq A_1, A_3 \leq 20.0\text{in.}^2$ and $0.002\text{in.}^2 \leq A_2 \leq 40.0\text{in.}^2$. The optimum design is obtained for a series of specified probabilities of occurrence ($p$). The expected value of the optimal weight and the final design variables, areas, are shown in Fig. 5.5. The mean value of weight is increased with increasing probability of success. The optimization results for the first- and second-order approximations in the stochastic analysis are very little different.

![Graph showing optimal weight versus probability](image)

(a) Optimal weight versus the probability of occurrence $p$.

Fig. 5.5 Optimal results of a fixed column.
Example 2: Propped cantilevered beam under a uniform load.

A propped cantilevered beam of length $l$ ($l = 100\text{in.}$) is subjected to a uniformly distributed load of intensity $w$ per unit length as shown in Fig. 5.6(a). The distributed load $w$ has a mean of 1.0 kip/in. and standard deviation of 0.316 kip/in. A width is 1 in. The Young’s modulus of the beam has two mean values, $\{\mu_E\}^T = [10000,10000]^{\text{ksi}}$. Their standard deviation and correlation coefficient matrix are as follows:

$$
\begin{bmatrix}
\sigma_{E_x} \\
\sigma_{E_2}
\end{bmatrix} = \begin{bmatrix} 700 \\
500 \end{bmatrix} \text{ksi} \quad \quad \quad \quad \begin{bmatrix} \rho_E \end{bmatrix} = \begin{bmatrix} 1.000 & 0.070 \\
0.070 & 1.000 \end{bmatrix}.
$$

The beam has for sizing design variables of two moments of inertia and two depths with mean values, standard deviation and correlation coefficient matrix as follows,
Therefore, the problem has seven stochastic variables defined by the normalized primitive random variables as follows,

\[ I_1 = \mu_1 \left(1 + q_{I_1}\right) = \mu_1 \left(1 + q_1\right) \]
\[ E_1 = \mu_{E_1} \left(1 + q_{E_1}\right) = \mu_5 \left(1 + q_5\right) \]
\[ I_2 = \mu_2 \left(1 + q_{I_2}\right) = \mu_2 \left(1 + q_2\right) \]
\[ E_2 = \mu_{E_2} \left(1 + q_{E_2}\right) = \mu_6 \left(1 + q_6\right) \]
\[ d_1 = \mu_{d_1} \left(1 + q_{d_1}\right) = \mu_3 \left(1 + q_3\right) \]
\[ w = \mu_w \left(1 + q_w\right) = \mu_7 \left(1 + q_7\right) \]
\[ d_2 = \mu_{d_2} \left(1 + q_{d_2}\right) = \mu_4 \left(1 + q_4\right) \]
The stochastic responses can be calculated as following:

$$
\begin{align*}
\text{Moment:} & \\
\begin{bmatrix}
\mu_{M_1} \\
\mu_{M_2} \\
\mu_{M_3} \\
\mu_{M_4}
\end{bmatrix} & = 
\begin{bmatrix}
1250.000 \\
-1875.000 \\
-1875.000 \\
0.000
\end{bmatrix} \text{ in.-k lb} \\
\begin{bmatrix}
\sigma_{M_1} \\
\sigma_{M_2} \\
\sigma_{M_3} \\
\sigma_{M_4}
\end{bmatrix} & = 
\begin{bmatrix}
46.790 \\
60.600 \\
60.600 \\
0.000
\end{bmatrix} \text{ in.-k lb} \\
\left[ \rho_M \right] & = 
\begin{bmatrix}
1.000 & -0.716 & -0.716 & 0.000 \\
-0.716 & 1.000 & 1.000 & 0.000 \\
-0.716 & 1.000 & 1.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000
\end{bmatrix}
\end{align*}
$$

Displacement/rotation:

$$
\begin{align*}
\begin{bmatrix}
\mu_{\theta_c} \\
\mu_{\theta_b} \\
\sigma_{\theta_c} \\
\sigma_{\theta_b}
\end{bmatrix} & = 
\begin{bmatrix}
-0.0006028 \text{ rad} \\
0.06028 \text{ in.} \\
0.0000799 \\
0.000158
\end{bmatrix} \\
\left[ \rho_\chi \right] & = 
\begin{bmatrix}
1.000 & -0.287 & 0.602 \\
-0.287 & 1.000 & -0.938 \\
0.602 & -0.938 & 1.000
\end{bmatrix}
\end{align*}
$$

Stress:

$$
\begin{align*}
\begin{bmatrix}
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{bmatrix} & = 
\begin{bmatrix}
8.681 \\
-13.021 \\
-13.021 \\
0.000
\end{bmatrix} \text{ ksi} \\
\begin{bmatrix}
\sigma_{\sigma_1} \\
\sigma_{\sigma_2} \\
\sigma_{\sigma_3} \\
\sigma_{\sigma_4}
\end{bmatrix} & = 
\begin{bmatrix}
0.708 \\
1.122 \\
1.079 \\
0.000
\end{bmatrix} \text{ ksi} \\
\left[ \rho_M \right] & = 
\begin{bmatrix}
1.000 & -0.951 & -0.302 & 0.000 \\
-0.951 & 1.000 & 0.232 & 0.000 \\
-0.302 & 0.232 & 1.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000
\end{bmatrix}
\end{align*}
$$

It is obvious that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of the first and second moments are about 3.7% at $A$ and 3.2% at $C$, respectively; the standard deviation of the displacement is 6.7% at $C$, the standard deviations of rotations are 13.3% at $C$ and 6.6% at $B$, respectively; the standard deviations of bending stresses are about 8%.
The probability density function and the cumulative distribution function for the bending moment $M_2$ at $C$ are shown in Fig. 5.7 (a) and (b), respectively. It should be noted that both the first- and second-order approximations have the almost same probabilistic description.

(a) Probability density functions for the bending moment $M_2$ at $C$.

(b) Cumulative distribution functions for the bending moment $M_2$ at $C$.

Fig. 5.7 Stochastic description of the bending moment at $C$. 
The calculated response values for different levels of occurrence (\( p = 50\%, 25\%, \) and \( 75\%) \) are listed in Table 5.2. From the table, it is noted that about two percent change is found in the first and second moments in the \( p = 25\% \) and \( 75\% \) levels; about 4.5% change is found in the displacement, about 9.0% and 4.6% changes are found in the rotations, whereas about 5.5% changes are noted in the two stresses.

Table 5.2 Response values for \( p\)-percent probability of success in the propped cantilevered beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p = 50% )</td>
</tr>
<tr>
<td>Moment</td>
<td>( p = 25% )</td>
</tr>
<tr>
<td>( M_1 ) (in.-klb)</td>
<td>1250.00</td>
</tr>
<tr>
<td>( M_2 ) (in.-klb)</td>
<td>1219.16 (97.53%)</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td>1282.28 (102.58%)</td>
</tr>
<tr>
<td>( \theta_C ) (rad)</td>
<td>-1875.00</td>
</tr>
<tr>
<td>( \nu_C ) (in.)</td>
<td>-1833.77 (97.80%)</td>
</tr>
<tr>
<td>( \theta_B ) (rad)</td>
<td>-1906.20 (101.66%)</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>( \sigma_1 ) (ksi)</td>
<td>0.0606</td>
</tr>
<tr>
<td>( \sigma_2 ) (ksi)</td>
<td>0.0579</td>
</tr>
<tr>
<td></td>
<td>-0.00242</td>
</tr>
<tr>
<td></td>
<td>-0.00232</td>
</tr>
<tr>
<td></td>
<td>8.704</td>
</tr>
<tr>
<td></td>
<td>8.226 (94.51%)</td>
</tr>
<tr>
<td></td>
<td>-13.053</td>
</tr>
<tr>
<td></td>
<td>-12.296 (94.20%)</td>
</tr>
<tr>
<td>Note: Lower value if mean is negative; Expressed as percent of mean.</td>
<td></td>
</tr>
</tbody>
</table>

The sensitivity analysis of moment at \( A \), displacement at \( C \) and bending stress at \( C \), with respect to the primitive random variables, is shown in Fig. 5.8 (a), (b) and (c) at the \( 75\% \) probability of occurrence level. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other. In the moment sensitivity, the moment at \( A \) is sensitive to the distributed load of intensity \( w \), not sensitive to two depths \( d_1 \) and \( d_2 \). The sensitivity of the displacement \( x_1 \) is similar to the sensitivity of the moment at \( A \). In the stress sensitivity, the bending stress at \( C \) is sensitive to the distributed load of intensity \( w \), moment of inertia \( I_1 \) and its depth \( d_1 \), not sensitive to the depth \( d_2 \).
Fig. 5.8 Sensitivity analysis of responses in the propped cantilevered beam.
For stochastic optimization, the material density has a mean of $0.289 \text{lbf/in.}^3$ and standard deviation of $0.005 \text{lbf/in.}^3$. The permissible stress has a mean of 8000 $\text{psi}$ and standard deviation of 800 $\text{psi}$. The displacement limitation has a mean of 0.06 $\text{in.}$ and standard deviation of 0.006 $\text{in.}$. Thus, the minimum expected weight and design variables are plotted in Fig. 5.9. The optimal results for the first- and second-order approximations in the stochastic analysis are almost equal.

![Optimal weight versus the probability of occurrence $p$.](image1)

(a) Optimal weight versus the probability of occurrence $p$.

![Design variables versus the probability of occurrence $p$.](image2)

(b) Design variables versus the probability of occurrence $p$.

Fig. 5.9 Optimal results of the propped cantilevered beam.
Example 3: Two-span beam under a uniform load.

A two-span beam of length $2l$ ($l = 50\text{ in.}$) is subjected to a uniformly distributed load of intensity $w$ per unit length, ($\mu_w = 15\text{kip/in.}$ and $\sigma_w = 1.5\text{kip/in.}$) as shown in Fig. 5.10(a). A width $b$ is 1 in. The Young’s modulus of the beam has two mean values, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{E_1} \\
\mu_{E_2}
\end{bmatrix} = \begin{bmatrix} 10,000 \\ 10,000 \end{bmatrix}_{\text{ksi}} \quad \begin{bmatrix}
\sigma_{E_1} \\
\sigma_{E_2}
\end{bmatrix} = \begin{bmatrix} 632.456 \\ 670.820 \end{bmatrix}_{\text{ksi}} \quad \begin{bmatrix}
\rho_E \\
\end{bmatrix} = \begin{bmatrix} 1.000 & 0.589 \\
0.589 & 1.000 \end{bmatrix}.
$$

![Diagram of a two-span beam under uniform load](image)

(a) Beam under load $w$.

![Diagram of forces in the beam](image)

(b) Forces acting in the beam.

Fig. 5.10 Two-span beam under uniform load.

Four sizing design variables in the beam are two moments of inertia and two depths with mean values, standard deviation and correlation coefficient matrix as follows:
Because of symmetrical structures, only half of beam is considered. Thus, the random variables are defined by seven normalized primitive random variables as

\[
\begin{align*}
I_1 &= \mu_{i_1} (1 + q_{i_1}) = \mu_i (1 + q_1) \\
I_2 &= \mu_{i_2} (1 + q_{i_2}) = \mu_i (1 + q_2) \\
d_1 &= \mu_{d_1} (1 + q_{d_1}) = \mu_d (1 + q_3) \\
d_2 &= \mu_{d_2} (1 + q_{d_2}) = \mu_d (1 + q_4)
\end{align*}
\]

The stochastic responses can be obtained as following:

\[
\begin{align*}
\text{Moment:} \quad & \begin{pmatrix} \mu_{\bar{M}_1} \\ \mu_{\bar{M}_2} \\ \mu_{\bar{M}_3} \\ \mu_{\bar{M}_4} \end{pmatrix} = \begin{pmatrix} \mu_{M_1} \\ \mu_{M_2} \\ \mu_{M_3} \\ \mu_{M_4} \end{pmatrix} - \begin{pmatrix} 0 \\ -703.1250 \\ -703.1250 \\ 4687.500 \end{pmatrix} \text{ in. - klb} \\
\text{Displacement/rotation:} \quad & \begin{pmatrix} \mu_{\bar{\theta}_1} \\ \mu_{\bar{\theta}_2} \\ \mu_{\bar{\theta}_3} \\ \mu_{\bar{\theta}_4} \end{pmatrix} = \begin{pmatrix} \mu_{\theta_1} \\ \mu_{\theta_2} \\ \mu_{\theta_3} \\ \mu_{\theta_4} \end{pmatrix} - \begin{pmatrix} 0.00452 \text{ rad} \\ 0.0065 \text{ in.} \\ 0.00113 \end{pmatrix} \text{ in. - klb} \\
\text{Stress:} \quad & \begin{pmatrix} \mu_{\bar{\sigma}_1} \\ \mu_{\bar{\sigma}_2} \\ \mu_{\bar{\sigma}_3} \\ \mu_{\bar{\sigma}_4} \end{pmatrix} = \begin{pmatrix} \mu_{\sigma_1} \\ \mu_{\sigma_2} \\ \mu_{\sigma_3} \\ \mu_{\sigma_4} \end{pmatrix} - \begin{pmatrix} 0 \\ -48.828 \\ -48.828 \\ 32.552 \end{pmatrix} \text{ ksi}
\end{align*}
\]
$$\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
5.034 \\
5.073 \\
3.388 \\
\end{bmatrix} ksi$$

$$[\rho_\Sigma] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1.000 & 0.993 & -0.986 \\
0 & 0.993 & 1.000 & -0.989 \\
0 & -0.986 & -0.989 & 1.000 \\
\end{bmatrix}$$

It may be noted that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of the second and fourth moments are about 10.0% at $D$ and $B$; the standard deviation of the displacement is 11.7% at $D$, the standard deviations of rotations are 11.7% at $A$ and 13.5% at $B$, respectively; the standard deviations of bending stresses are about 10%.

The probability density function and the cumulative distribution function for the displacement at $D$ are shown in Fig. 5.11(a) and (b), respectively. It is obvious that both the first- and second-order approximations have a small difference in the probabilistic description.

(a) Probability density functions for the displacement at $D$.

Fig. 5.11 Stochastic description of the displacement at $D$. 
(b) Cumulative distribution functions for the displacement at $D$.

Fig. 5.11 Stochastic description of the displacement at $D$ (Continued).

The calculated response values for different levels of occurrence ($p = 50\%, 25\%$ and $75\%$) are listed in Table 5.3. In the table, we note that about seven percent changes are found in the second and fourth moments in the $p = 25\%$ and $75\%$ levels; about 8% change is found in the displacement, about 7.9% and 9.7% changes are found in the rotations, whereas about 6.0% to 7.0% changes are noted in the two stresses.

Table 5.3 Response values for $p$-percent probability of success in the two-span beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>$p = 50%$</th>
<th>$p = 25%$</th>
<th>$p = 75%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_2$ (in.-klb)</td>
<td>-7030.772</td>
<td>-6556.179</td>
<td>-7505.365</td>
<td>(106.75%)</td>
</tr>
<tr>
<td>$M_4$ (in.-klb)</td>
<td>4688.456</td>
<td>4370.239</td>
<td>5006.673</td>
<td>(106.79%)</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_A$ (rad)</td>
<td>-0.00454</td>
<td>-0.00418</td>
<td>-0.00490</td>
<td>(107.93%)</td>
</tr>
<tr>
<td>$v_D$ (in.)</td>
<td>0.0568</td>
<td>0.0523</td>
<td>0.0612</td>
<td>(107.75%)</td>
</tr>
<tr>
<td>$\theta_B$ (rad)</td>
<td>0.00113</td>
<td>0.00103</td>
<td>0.00124</td>
<td>(109.73%)</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_1$ (ksi)</td>
<td>-48.839</td>
<td>-45.444</td>
<td>-52.235</td>
<td>(105.95%)</td>
</tr>
<tr>
<td>$\sigma_2$ (ksi)</td>
<td>32.571</td>
<td>30.286</td>
<td>34.856</td>
<td>(107.02%)</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.
The sensitivity analysis of moment, displacement and bending stress at $D$, with respect to the random variables, is shown in Fig. 5.12(a), (b) and (c) at the 75% probability of occurrence. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Most of stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $D$ is sensitive to the distributed load of intensity $w$, not sensitive to two depths $d_1$ and $d_2$. The sensitivity of the displacement $x_1$ is sensitive to all random variables except for two depths $d_1$ and $d_2$. In the stress sensitivity, the bending stress at $D$ is sensitive to the distributed load of intensity $w$, moment of inertia $I_2$ and its depth $d_2$, not sensitive to the depth $d_1$.

(a) Sensitivity for moment at $D$.

Fig. 5.12 Sensitivity analysis of responses in the two-span beam.
Fig. 5.12 Sensitivity analysis of responses in the two-span beam (Continued).

In addition, the material density is assumed to have a mean value of 0.289 $\text{lbf/in.}^3$ and standard derivation of 0.005 $\text{lbf/in.}^3$. The allowable strength has a mean of 40,000 $\text{psi}$ and standard deviation of 4000 $\text{psi}$. The displacement limitation has a mean of 0.05 $\text{in.}$ and standard deviation of 0.005 $\text{in.}$ and $0.001\text{in.} \leq d_1, d_2 \leq 24.0\text{in.}$ As a result, the optimal expected weight and design variables of beam for different probability of occurrence $p$ are shown in Fig. 5.13. The optimization results for the first- and second-order approximations in the stochastic analysis are almost equal, since the variances are small.
Example 4: Continuous beam with mechanical, thermal and support settling loads.

A two-span continuous beam, shown in Fig.5.14, has six random material variables that consist of three Young’s modulus, $E_1, E_2, E_3$ and three coefficients of thermal expansions, $\alpha_1, \alpha_2, \alpha_3$. The stochastic properties of these variables are as follows:
There is a concentrated load at the center of the second span that has a mean of 
\[ \frac{\mu}{\mu} \] and standard deviation of 1.0
\[ \frac{\mu}{\mu} \] kip. The center support settles with a mean of 0.25
\[ \frac{\mu}{\mu} \] in. and a standard deviation of 0.05 in. The temperature of the upper and lower fibers of
\[ \frac{\mu}{\mu} \] the first span of the beam has the following stochastic properties:

\[ \begin{bmatrix} \mu_{i_1} \\ \mu_{i_2} \\ \mu_{i_3} \\ \mu_{d_1} \\ \mu_{d_2} \\ \mu_{d_3} \end{bmatrix} = \begin{bmatrix} 864 \text{in.}^4 \\ 864 \text{in.}^4 \\ 864 \text{in.}^4 \\ 12 \text{in.} \\ 12 \text{in.} \\ 12 \text{in.} \end{bmatrix}, \begin{bmatrix} \sigma_{i_1} \\ \sigma_{i_2} \\ \sigma_{i_3} \\ \sigma_{d_1} \\ \sigma_{d_2} \\ \sigma_{d_3} \end{bmatrix} = \begin{bmatrix} 43.2 \text{in.}^3 \\ 43.2 \text{in.}^3 \\ 43.2 \text{in.}^3 \\ 0.720 \text{in.} \\ 0.720 \text{in.} \\ 0.720 \text{in.} \end{bmatrix} \]

\[ \begin{bmatrix} \rho \end{bmatrix} = \begin{bmatrix} 1.00 & 0.10 & 0.10 & 0 & 0 & 0 \\ 0.10 & 1.00 & 0.10 & 0 & 0 & 0 \\ 0.10 & 0.10 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.00 & 0.10 & 0.10 \\ 0 & 0 & 0 & 0.10 & 1.00 & 0.10 \\ 0 & 0 & 0 & 0.10 & 0.10 & 1.00 \end{bmatrix} \]

Thus, the problem has sixteen stochastic variables defined as:

\[ I_1 = \mu_{i_1}(1 + q_{i_1}) = \mu_i(1 + q_i) \]
\[ I_2 = \mu_{i_2}(1 + q_{i_2}) = \mu_2(1 + q_2) \]
\[ I_3 = \mu_{i_3}(1 + q_{i_3}) = \mu_3(1 + q_3) \]
\[ d_1 = \mu_{d_1}(1 + q_{d_1}) = \mu_d(1 + q_d) \]
\[ d_2 = \mu_{d_2}(1 + q_{d_2}) = \mu_5(1 + q_5) \]
\[ E_3 = \mu_{E_1}(1 + q_{E_3}) = \mu_9(1 + q_9) \]
\[ \alpha_1 = \mu_{\alpha_1}(1 + q_{\alpha_1}) = \mu_{\alpha_1}(1 + q_{\alpha_1}) \]
\[ \alpha_2 = \mu_{\alpha_2}(1 + q_{\alpha_2}) = \mu_{\alpha_2}(1 + q_{\alpha_2}) \]
\[ \alpha_3 = \mu_{\alpha_3}(1 + q_{\alpha_3}) = \mu_{\alpha_3}(1 + q_{\alpha_3}) \]
\[ P = \mu_p(1 + q_p) = \mu_3(1 + q_{33}) \]
\begin{align*}
d_3 &= \mu_{d_3} (1 + q_{d_3}) = \mu_c (1 + q_c) \\
E_1 &= \mu_{E_1} (1 + q_{E_1}) = \mu_\gamma (1 + q_\gamma) \\
E_2 &= \mu_{E_2} (1 + q_{E_2}) = \mu_\delta (1 + q_\delta) \\
T_u &= \mu_{T_u} (1 + q_{T_u}) = \mu_{t_4} (1 + q_{t_4}) \\
T_l &= \mu_{T_l} (1 + q_{T_l}) = \mu_{t_5} (1 + q_{t_5}) \\
\Delta &= \mu_{\Delta} (1 + q_\Delta) = \mu_{t_6} (1 + q_{t_6})
\end{align*}

(a) Continuous beam under loads.

(b) Analysis model.

Fig.5.14 Two-span beam with mechanical, thermal and support settling loads.

The stochastic responses of the selected six variables are as follows:

\[
\begin{bmatrix}
\mu_{\bar{M}_2} \\
\mu_{\bar{M}_4} \\
\mu_{\bar{c}} \\
\mu_{\bar{\theta}_b} \\
\mu_{\bar{\sigma}_b} \\
\mu_{\bar{\sigma}_c}
\end{bmatrix} =
\begin{bmatrix}
\mu_{M_2} \\
\mu_{M_4} \\
\mu_{c} \\
\mu_{\theta_b} \\
\mu_{\sigma_b} \\
\mu_{\sigma_c}
\end{bmatrix}
\begin{bmatrix}
242.100in. - klb \\
-478.950in. - klb \\
0.357in. \\
0.000883rad \\
1.681ksi \\
-3.326ksi
\end{bmatrix}
\begin{bmatrix}
\mu_{M_2} \\
\mu_{M_4} \\
\mu_{c} \\
\mu_{\theta_b} \\
\mu_{\sigma_b} \\
\mu_{\sigma_c}
\end{bmatrix}
\begin{bmatrix}
242.732in. - klb \\
-478.634in. - klb \\
0.359in. \\
0.000890rad \\
1.685ksi \\
-3.332ksi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{M_2} \\
\sigma_{M_4} \\
\sigma_{c} \\
\sigma_{\theta_b} \\
\sigma_{\sigma_b} \\
\sigma_{\sigma_c}
\end{bmatrix} =
\begin{bmatrix}
37.054in. - klb \\
50.924in. - klb \\
0.0450in. \\
0.000288rad \\
0.257ksi \\
0.437ksi
\end{bmatrix}
\begin{bmatrix}
1.000 \\
-0.352 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
1.000 \\
-0.352 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
[p_{rr}] =
\begin{bmatrix}
1.000 & -0.352 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1.000 \\
-0.352 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0.598 & 1.000 & 0.598 & 0 & 0 & 0 \\
0.598 & 1.000 & 0.598 & 0 & 0 & 0 \\
1.000 & -0.371 & 1.000 & -0.371 & 1.000 & 0.371
\end{bmatrix}
\]
It is obvious that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of the second and fourth moments are about 15.3% at $B$ and 10.6% at $C$, respectively; the standard deviation of the displacement is 12.6% at $C$, the standard deviation of rotation is 32.6% at $B$; the standard deviations of bending stresses are about 15.3% at $B$ and 13.1% at $C$.

The probability density function and the cumulative distribution function for the bending stress at $C$ are shown in Fig. 5.15(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description are almost equal to each other.

(a) Probability density function for the bending stress at $C$.

Fig.5.15 Stochastic description of the bending stress at $C$. 
(b) Cumulative distribution function for the bending stress at C.

Fig.5.15 Stochastic description of the bending stress at C (Continued).

The selected response values for different probability levels of occurrence ($p = 50\%, 25\%$ and $75\%$) for moments, bending stresses, and displacement are listed in Table 5.4. In the table, we note that about ten and seven percent changes are found in the second and fourth moments in the $p = 25\%$ and $75\%$ levels; about $8.4\%$ change is found in the displacement at $C$, about $21.8\%$ change is found in the rotation at $B$, whereas about $8.9\%$ to $10.3\%$ changes are noted in the two bending stresses.

Table 5.4 Response values for p-percent probability of success in the continuous beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>$p = 50%$</th>
<th>$p = 25%$</th>
<th>$p = 75%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_2$ (in.-klb)</td>
<td>242.732</td>
<td>217.740 (89.70%)</td>
<td>267.725 (110.30%)</td>
<td></td>
</tr>
<tr>
<td>$M_4$ (in.-klb)</td>
<td>-478.634</td>
<td>-444.286 (92.82%)</td>
<td>-512.982 (107.18%)</td>
<td></td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_B$ (rad)</td>
<td>0.000890</td>
<td>0.000696 (78.20%)</td>
<td>0.001084 (121.80%)</td>
<td></td>
</tr>
<tr>
<td>$v_C$ (in.)</td>
<td>0.359</td>
<td>0.328 (91.36%)</td>
<td>0.389 (108.36%)</td>
<td></td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_B$ (ksi)</td>
<td>1.685</td>
<td>1.511 (89.67%)</td>
<td>1.858 (110.27%)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_C$ (ksi)</td>
<td>-3.332</td>
<td>-3.037 (91.15%)</td>
<td>-3.627 (108.85%)</td>
<td></td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.
The sensitivity at the 75% probability of occurrence level, with respect to the primitive random variables, is shown in Fig. 5.16(a), (b) and (c) for the moment at C, the displacement at C and bending stress at B, respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at C is sensitive to the load $P$, not sensitive to the depths $d_2$ and $d_3$ and coefficients of thermal expansion $\alpha_2$ and $\alpha_3$. The sensitivity of the displacement $x_c$ is sensitive to the loads $P$ and the center settling support $\Delta$, not sensitive to the depths $d_2$ and $d_3$ and coefficients of thermal expansion $\alpha_2$ and $\alpha_3$. In the stress sensitivity, the bending stress at B is sensitive to the moment of inertia $I_2$, depth $d_2$ and the load $P$, not sensitive to the depth $d_3$ and coefficients of thermal expansion $\alpha_2$ and $\alpha_3$.

(a) Sensitivity for moment at C.

Fig. 5.16 Sensitivity analysis of responses in the continuous beam.
For stochastic optimization, it is assumed that the material density has three mean values, standard deviation and correlation coefficient matrix given by

\[
\begin{align*}
\mu_\rho_1 &= 0.289 \text{ lbf/in}^3, & \sigma_\rho_1 &= 0.0289 \text{ lbf/in}^3, & \rho_\rho &= \begin{bmatrix} 1.000 & 0.502 & 0.248 \\ 0.502 & 1.000 & 0.248 \\ 0.248 & 0.248 & 1.000 \end{bmatrix}, \\
\mu_\rho_2 &= 0.289 \text{ lbf/in}^3, & \sigma_\rho_2 &= 0.0145 \text{ lbf/in}^3. \\
\mu_\rho_3 &= 0.289 \text{ lbf/in}^3.
\end{align*}
\]

The permissible stress has a mean of 2.5 ksi and standard deviation of 0.15 ksi. The displacement limitation has a mean of 0.3 in. and standard deviation of 0.0015 in. and 0.01 in. \(d_1, d_2, d_3 \leq 24.0 \text{in.}\) The optimal expected weight and final design variables
of the beam for different probability of occurrence $p$ are shown in Fig. 5.17. The optimization results for both the first- and second-order approximations in the stochastic analysis are almost equal, since the variances are small.

(a) Optimal weight versus the probability of occurrence $p$.

(b) Design variables versus the probability of occurrence $p$.

Fig. 5.17 Optimal results of the continuous beam.
Example 5: Propped beam for a mechanical, thermal and support settling loads.

A uniform propped beam shown in Fig. 5.18 of length $2a$ ($a = 120\text{in.}$) has two random moments of inertia, $I_1$ and $I_2$, and two depths, $d_1$ and $d_2$. Their mean values, standard deviation and correlation coefficient matrix are as follows:

$$\begin{align*}
\mu_{I_1} &= \begin{bmatrix} 100\text{in.}^4 \\ 100\text{in.}^4 \end{bmatrix}, \\
\mu_{I_2} &= \begin{bmatrix} 100\text{in.}^4 \\ 100\text{in.}^4 \end{bmatrix}, \\
\mu_{d_1} &= \begin{bmatrix} 10\text{in.} \\ 10\text{in.} \end{bmatrix}, \\
\mu_{d_2} &= \begin{bmatrix} 10\text{in.} \\ 10\text{in.} \end{bmatrix}, \\
\sigma_{I_1} &= \begin{bmatrix} 5000\text{in.}^4 \\ 2236\text{in.}^4 \end{bmatrix}, \\
\sigma_{I_2} &= \begin{bmatrix} 5000\text{in.}^4 \\ 2236\text{in.}^4 \end{bmatrix}, \\
\sigma_{d_1} &= \begin{bmatrix} 0.6\text{in.} \\ 0.6\text{in.} \end{bmatrix}, \\
\sigma_{d_2} &= \begin{bmatrix} 0.6\text{in.} \\ 0.6\text{in.} \end{bmatrix}, \\
\rho_{d_1d_2} &= \begin{bmatrix} 1.000 & 0.224 & 0 & 0 \\ 0.224 & 1.000 & 0 & 0 \\ 0 & 0 & 1.000 & 0.1 \\ 0 & 0 & 0.1 & 1.000 \end{bmatrix}.
\end{align*}$$

(a) Propped beam under loads.

(b) Analysis model.

Fig. 5.18 Propped beam with mechanical thermal and settling support loads.
The width, $b$, is 1.2 in. Two stochastic material variables consist of a Young’s modulus, $E$, and a coefficient of thermal expansion, $\alpha$. Their stochastic properties of two variables are as follows:

\[
\begin{bmatrix}
\mu_E \\
\sigma_E
\end{bmatrix} = \begin{bmatrix}
30,000 ksi \\
2100 ksi
\end{bmatrix} \\
\begin{bmatrix}
\mu_\alpha \\
\sigma_\alpha
\end{bmatrix} = \begin{bmatrix}
1.2 \times 10^{-5} F \\
3.286 \times 10^{-7} F
\end{bmatrix} \\
\begin{bmatrix}
\rho_{mv}
\end{bmatrix} = \begin{bmatrix}
1.00 & 0.128 \\
0.128 & 1.00
\end{bmatrix}.
\]

The mechanical load, $P$, has a mean of 10.0 kip and standard deviation of 1.0 kip. The support settles with a mean, $\mu_\Delta$, of 0.25 in. and a standard deviation, $\sigma_\Delta$, of 0.05 in. Temperature is assumed to be uniform along the length of the beam. Along the depth, the temperature variation is linear, with values, $T_u$ and $T_l$ at the upper and lower surfaces, as shown in Fig.5.18(a). The temperatures, $T_u$ and $T_l$, have the following stochastic properties:

\[
\begin{bmatrix}
\mu_{T_u} \\
\sigma_{T_u}
\end{bmatrix} = \begin{bmatrix}
10 \\
1.2
\end{bmatrix} \, F \\
\begin{bmatrix}
\mu_{T_l} \\
\sigma_{T_l}
\end{bmatrix} = \begin{bmatrix}
10 \\
1.2
\end{bmatrix} \, F \\
\begin{bmatrix}
\rho_T
\end{bmatrix} = \begin{bmatrix}
1.00 & -0.10 \\
-0.10 & 1.00
\end{bmatrix}.
\]

Therefore, the problem had ten random variables defined as:

\[
\begin{align*}
I_1 &= \mu_{I_1} \left(1 + q_{I_1}\right) = \mu_1 \left(1 + q_1\right) & \alpha &= \mu_\alpha \left(1 + q_\alpha\right) = \mu_\alpha \left(1 + q_\alpha\right) \\
I_2 &= \mu_{I_2} \left(1 + q_{I_2}\right) = \mu_2 \left(1 + q_2\right) & P &= \mu_P \left(1 + q_P\right) = \mu_P \left(1 + q_P\right) \\
d_1 &= \mu_{d_1} \left(1 + q_{d_1}\right) = \mu_3 \left(1 + q_3\right) & T_u &= \mu_{T_u} \left(1 + q_{T_u}\right) = \mu_8 \left(1 + q_8\right) \\
d_2 &= \mu_{d_2} \left(1 + q_{d_2}\right) = \mu_4 \left(1 + q_4\right) & T_l &= \mu_{T_l} \left(1 + q_{T_l}\right) = \mu_9 \left(1 + q_9\right) \\
E &= \mu_E \left(1 + q_E\right) = \mu_5 \left(1 + q_5\right) & \Delta &= \mu_\Delta \left(1 + q_\Delta\right) = \mu_{10} \left(1 + q_{10}\right)
\end{align*}
\]

The stochastic responses can be obtained as followings:

\[
\begin{align*}
\text{Moment:} & \quad \begin{bmatrix}
\mu_{M_1} \\
\mu_{M_2} \\
\mu_{M_3} \\
\mu_{M_4}
\end{bmatrix} = \begin{bmatrix}
0 \\
-301.469 \\
-301.469 \\
597.063
\end{bmatrix} \, in.-k lb \\
\text{Torque:} & \quad \begin{bmatrix}
\mu_{M_1} \\
\mu_{M_2} \\
\mu_{M_3} \\
\mu_{M_4}
\end{bmatrix} = \begin{bmatrix}
0 \\
-301.190 \\
-301.190 \\
597.063
\end{bmatrix} \, in.-k lb.
\end{align*}
\]
Displacement/rotation:

\[
\begin{align*}
\{\sigma_{M_1}\} &= \begin{bmatrix} 0 \\ 38.598 \\ 38.598 \\ 48.573 \end{bmatrix} \text{ in. - klb} \\
\{\sigma_{M_2}\} &= \begin{bmatrix} 118 \end{bmatrix} \\
\{\sigma_{M_3}\} &= \begin{bmatrix} 118 \end{bmatrix} \\
\{\sigma_{M_4}\} &= \begin{bmatrix} 118 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix} \mu_{\theta_3} \\ \mu_{\theta_4} \end{bmatrix} = \begin{bmatrix} -0.00588 \text{ rad} \\ -0.541 \text{ in.} \end{bmatrix} \\
\begin{bmatrix} \mu_{\theta_1} \\ \mu_{\theta_2} \end{bmatrix} = \begin{bmatrix} 0.000303 \text{ rad} \\ 0.00304 \text{ rad} \end{bmatrix} \\
\begin{bmatrix} \sigma_{\theta_3} \\ \sigma_{\theta_4} \end{bmatrix} = \begin{bmatrix} 0.000831 \text{ rad} \\ 0.0548 \text{ in.} \end{bmatrix} \\
\begin{bmatrix} \sigma_{\theta_1} \\ \sigma_{\theta_2} \end{bmatrix} = \begin{bmatrix} 0.000323 \text{ rad} \\ -0.00592 \text{ rad} \end{bmatrix}
\]

\[
\begin{bmatrix} \sigma_{\tau_1} \\ \sigma_{\tau_2} \end{bmatrix} = \begin{bmatrix} 2.260 \\ 2.225 \end{bmatrix} \text{ ksi} \\
\begin{bmatrix} \sigma_{\tau_3} \\ \sigma_{\tau_4} \end{bmatrix} = \begin{bmatrix} 2.894 \\ 2.260 \end{bmatrix} \text{ ksi}
\]

\[
\begin{bmatrix} \rho_M \end{bmatrix} = \begin{bmatrix} 0.1000 & 1.000 & 0.00 \end{bmatrix} \begin{bmatrix} 0.0000 & 1.000 & -0.811 \end{bmatrix} \\
\begin{bmatrix} \rho_X \end{bmatrix} = \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix} \\
\begin{bmatrix} \rho_Y \end{bmatrix} = \begin{bmatrix} 0.0548 \end{bmatrix} \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix} \\
\begin{bmatrix} \rho_{\theta_1} \end{bmatrix} = \begin{bmatrix} 0.000323 \end{bmatrix} \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix} \\
\begin{bmatrix} \rho_{\theta_2} \end{bmatrix} = \begin{bmatrix} 0.000831 \end{bmatrix} \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix} \\
\begin{bmatrix} \rho_{\theta_3} \end{bmatrix} = \begin{bmatrix} 0.000323 \end{bmatrix} \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix} \\
\begin{bmatrix} \rho_{\theta_4} \end{bmatrix} = \begin{bmatrix} 0.000831 \end{bmatrix} \begin{bmatrix} 1.000 \end{bmatrix} \begin{bmatrix} 0.772 & 1.000 & -0.800 \end{bmatrix}
\]

It may be noted that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of the second and fourth moments are about 12.8% at B and 8.1% at C, respectively; the standard deviation of the displacement is 10.1% at B, the standard deviations of rotations are 14.1% at A and 10.7% at B, respectively; the standard deviations of bending stresses are about 15.0% and 9.7%, respectively.

The probability density function and cumulative distribution function for moment at B are shown in Fig. 5.19 (a) and (b), respectively. It is obvious that both the first- and
second-order approximations in the probabilistic description are almost equal to each other.

![Graph of probability density functions for moment at B.](image)

(a) Probability density functions for moment at $B$.

![Graph of cumulative distribution functions for moment at B.](image)

(b) Cumulative distribution functions for moment at $B$.

Fig. 5.19 Stochastic description of the moment at $B$.

The calculated response variable for different probability level of occurrence ($p = 50\%, 25\%$ and $75\%$) are listed in Table 5.5. In the table, we note that about $8.6\%$ and $5.5\%$ percent changes are found in the second and fourth moments in the $p = 25\%$ and
75% levels; about 6.8% change is found in the displacement at \( B \), about 9.5% and 7.2% changes is found in the rotation at \( A \) and \( B \), whereas about 10.1% to 6.5% changes are noted in the two bending stresses.

Table 5.5 Response values for \( p \)-percent probability of success in the propped beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p = 50% )</td>
</tr>
<tr>
<td>Moment</td>
<td></td>
</tr>
<tr>
<td>( M_2 ) (in.-klb)</td>
<td>-301.190</td>
</tr>
<tr>
<td>( M_4 ) (in.-klb)</td>
<td>597.619</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
</tr>
<tr>
<td>( \theta_A ) (rad)</td>
<td>-0.00592</td>
</tr>
<tr>
<td>( v_B ) (in.)</td>
<td>-0.544</td>
</tr>
<tr>
<td>( \theta_B ) (rad)</td>
<td>0.00304</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>( \sigma_2 ) (ksi)</td>
<td>-15.097</td>
</tr>
<tr>
<td>( \sigma_4 ) (ksi)</td>
<td>29.876</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of moment at \( B \), displacement at \( B \) and stress at \( C \) with respect to the random variables at 75% probability of occurrence is shown in Fig. 5.20 (a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at \( B \) is sensitive to the load \( P \) and moment of inertia \( I_2 \), the depths \( d_i \) has little effect on the moment at \( B \). The sensitivity of the displacement \( x_0 \) is sensitive to the loads \( P \), Young’s modulus, \( E \) and moment of inertia \( I_2 \), not sensitive to the depths \( d_2 \). In the stress sensitivity, the bending stress at \( C \) is sensitive to the depth \( d_2 \), moment of inertia \( I_2 \), and the load \( P \), not sensitive to the depth \( d_1 \) and the settling support \( \Delta \).
Fig. 5.20 Sensitivity analysis of responses in the propped beam.
In addition, it is assumed that the material density has a mean of 0.289 lbf/in.\(^3\) and standard deviation of 0.005 lbf/in.\(^3\). The allowable strength has a mean of 16,000 psi and standard deviation 1600 psi. The allowable displacement has a mean of 0.5 in. and standard deviation of 0.05 in. and 0.01 in.\( \leq d_1, d_2 \leq 30.0\) in. As a result, the optimal weight and design variables of beam are shown in Fig. 5.21. The optimization results for both the first- and second-order approximations in the stochastic analysis are almost equal.

![Graph](image_url)

(a) Optimal weight versus the probability of occurrence \(p\).

![Graph](image_url)

(b) Design variables versus the probability of occurrence \(p\).

Fig. 5.21 Optimal results of the propped beam.
Example 6: Fixed beam under a uniform load, thermal load and support settling loads.

A uniform beam of length \( l \) (\( l = 50\text{in.} \)) is fixed at both ends as shown in Fig. 5.22(a). Four sizing design variables consist of two moments of inertia \( I_1 \) and \( I_2 \), and two depths, \( d_1 \) and \( d_2 \) for beam spans (AC and CB). Their stochastic properties are as follows:

\[
\begin{bmatrix}
\mu_{I_1} \\
\mu_{I_2} \\
\mu_{d_1} \\
\mu_{d_2}
\end{bmatrix} = \begin{bmatrix}
864\text{in.}^4 \\
864\text{in.}^4 \\
12\text{in.} \\
12\text{in.}
\end{bmatrix}
\begin{bmatrix}
\sigma_{I_1} \\
\sigma_{I_2} \\
\sigma_{d_1} \\
\sigma_{d_2}
\end{bmatrix} = \begin{bmatrix}
14.965\text{in.}^4 \\
54.644\text{in.}^4 \\
0.228\text{in.} \\
0.228\text{in.}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.000 & 0.228 & 0 & 0 \\
0.228 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 1.000 \\
0 & 0 & 1.000 & 1.000
\end{bmatrix}
\]

(a) Beam fixed at both ends.

(b) free-body diagram.

Fig. 5.22 Fixed beam under uniform loads.

The stochastic properties of two material variables Young’s modulus, \( E \), and coefficient of thermal expansion, \( \alpha \), have the following mean value, standard deviation and correlation coefficient matrix:
\[
\begin{align*}
\{\mu_E\} &= \begin{bmatrix} 10,000 \text{ ksi} \\ 1.2 \times 10^{-5} \text{ ksi/F} \end{bmatrix}, \\
\{\sigma_E\} &= \begin{bmatrix} 187.083 \text{ ksi} \\ 2.546 \times 10^{-7} \text{ ksi/F} \end{bmatrix}, \\
\rho_{\text{ms}} &= \begin{bmatrix} 1.00 & 0.630 \\ 0.630 & 1.000 \end{bmatrix}.
\end{align*}
\]

The width, \(b\), is 6 in. the mechanical load \(w\) has a mean of 15.0 kip/in. and standard deviation of 1.5 kip/in. Temperature is assumed to be uniform along the length of the beam. Along the depth, the temperature variation is linear with values \(T_u\) and \(T_l\) at the upper and lower surfaces, as shown in Fig. 5.22(a). The temperatures, \(T_u\) and \(T_l\) have the following mean, standard deviation and correlation coefficient matrix as:

\[
\begin{align*}
\{\mu_{T_u}\} &= \begin{bmatrix} -10 \\ 10 \end{bmatrix} \text{ F}, \\
\{\sigma_{T_u}\} &= \begin{bmatrix} 0.500 \\ 0.707 \end{bmatrix} \text{ F}, \\
\rho_T &= \begin{bmatrix} 1.00 & -1.131 \\ -1.131 & 1.000 \end{bmatrix}.
\end{align*}
\]

The support settles at ends \(A\) and \(B\) with the following stochastic properties:

\[
\begin{align*}
\{\mu_{\Delta_A}\} &= \begin{bmatrix} -0.25 \\ -0.15 \end{bmatrix} \text{ in.}, \\
\{\mu_{\Delta_B}\} &= \begin{bmatrix} 0.0559 \\ 0.0382 \end{bmatrix} \text{ in.}, \\
\rho_{\Delta} &= \begin{bmatrix} 1.00 & 0.790 \\ 0.790 & 1.000 \end{bmatrix}.
\end{align*}
\]

Thus, the problem has eleven random variables defined as follows:

\[
\begin{align*}
I_1 &= \mu_{I_1} \left(1 + q_{I_1}\right) = \mu_1 \left(1 + q_1\right) \\
I_2 &= \mu_{I_2} \left(1 + q_{I_2}\right) = \mu_2 \left(1 + q_2\right) \\
d_1 &= \mu_{d_1} \left(1 + q_{d_1}\right) = \mu_3 \left(1 + q_3\right) \\
d_2 &= \mu_{d_2} \left(1 + q_{d_2}\right) = \mu_4 \left(1 + q_4\right) \\
E &= \mu_E \left(1 + q_E\right) = \mu_5 \left(1 + q_5\right) \\
\alpha &= \mu_\alpha \left(1 + q_\alpha\right) = \mu_6 \left(1 + q_6\right) \\
w &= \mu_w \left(1 + q_w\right) = \mu_7 \left(1 + q_7\right) \\
T_u &= \mu_{T_u} \left(1 + q_{T_u}\right) = \mu_8 \left(1 + q_8\right) \\
T_l &= \mu_{T_l} \left(1 + q_{T_l}\right) = \mu_9 \left(1 + q_9\right) \\
\Delta_A &= \mu_{\Delta_A} \left(1 + q_{\Delta_A}\right) = \mu_{10} \left(1 + q_{10}\right) \\
\Delta_B &= \mu_{\Delta_B} \left(1 + q_{\Delta_B}\right) = \mu_{11} \left(1 + q_{11}\right)
\end{align*}
\]

The stochastic responses can be obtained as follows:

\[
\begin{align*}
\left\{\mu_{M_1}\right\} &= \begin{bmatrix} 12327.20 \\ -24654.40 \end{bmatrix} \text{ in. - k lb}, \\
\left\{\mu_{M_2}\right\} &= \begin{bmatrix} 12334.014 \\ -24652.045 \end{bmatrix} \text{ in. - k lb}.
\end{align*}
\]
\[
\begin{align*}
\sigma_{M_1} &= 1272.984 \text{ in.-k}l b \\
\sigma_{M_2} &= 2500.207 \\
\sigma_{M_3} &= 2500.207 \\
\sigma_{M_4} &= 1278.304
\end{align*}
\]

\[
\begin{bmatrix}
\rho_M
\end{bmatrix} =
\begin{bmatrix}
1.000 & -0.983 & -0.983 & -0.922 \\
-0.983 & 1.000 & 1.000 & -0.976 \\
-0.983 & 1.000 & 1.000 & -0.976 \\
0.922 & -0.976 & -0.976 & 1.000
\end{bmatrix}.
\]

Displacement/rotation:

\[
\begin{align*}
\mu_{\nu_C} &= \begin{bmatrix} \mu_{\nu_1} \\ \mu_{\nu_2} \end{bmatrix} = \begin{bmatrix} 0.652 \text{ in.} \\ 0.00150 \text{ rad} \end{bmatrix} \\
\mu_{\theta_C} &= \begin{bmatrix} \mu_{\theta_1} \\ \mu_{\theta_2} \end{bmatrix} = \begin{bmatrix} 0.653 \text{ in.} \\ 0.00148 \text{ rad} \end{bmatrix} \\
\sigma_{\nu_C} &= 0.0660\text{ in.} \\
\sigma_{\theta_C} &= 0.000659\text{ rad}
\end{align*}
\]

\[
\begin{bmatrix}
\rho_X
\end{bmatrix} =
\begin{bmatrix}
1.000 & 0.157 \\
0.157 & 1.000
\end{bmatrix}.
\]

Stress:

\[
\begin{align*}
\mu_{\sigma_1} &= \begin{bmatrix} \mu_{\sigma_1} \\ \mu_{\sigma_2} \end{bmatrix} = \begin{bmatrix} 85.606 \\ -171.211 \end{bmatrix} \text{ ksi} \\
\mu_{\sigma_3} &= \begin{bmatrix} \mu_{\sigma_3} \\ \mu_{\sigma_4} \end{bmatrix} = \begin{bmatrix} -171.211 \\ 92.806 \end{bmatrix} \\
\sigma_{\sigma_1} &= \begin{bmatrix} 9.097 \\ 17.925 \\ 20.796 \\ 10.031 \end{bmatrix} \text{ ksi} \\
\sigma_{\sigma_2} &= \begin{bmatrix} -0.984 \\ 1.000 \\ 0.862 \\ -0.894 \end{bmatrix} \\
\sigma_{\sigma_3} &= \begin{bmatrix} -0.905 \\ 0.862 \\ 1.000 \\ -0.988 \end{bmatrix} \\
\sigma_{\sigma_4} &= \begin{bmatrix} 0.912 \\ -0.894 \\ -0.988 \\ 1.000 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
\rho_M
\end{bmatrix} =
\begin{bmatrix}
1.000 & -0.984 & -0.905 & 0.912 \\
-0.984 & 1.000 & 0.862 & -0.894 \\
-0.905 & 0.862 & 1.000 & -0.988 \\
0.912 & -0.894 & -0.988 & 1.000
\end{bmatrix}.
\]

Note that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of three moments are about 10.3% at \(A\), 10.1% at \(C\) and 9.6% at \(B\), respectively; the standard deviation of the displacement is 10.1% at \(C\), the standard deviation of rotation are 43.9% at \(C\); the standard deviations of bending stresses are about 10.6%, 10.5% and 10.8%, respectively.

The probability density function and cumulative distribution function for the displacement at \(C\) are shown in Fig. 5.23 (a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description are almost equal to each other.
The calculated response variable for different probability level of occurrence \( p = 50\%, 25\% \) and 75\% \) are listed in Table 5.6. In the table, it may be noted that about 6.5\% to 7.0\% percent changes are found in the three moments in the \( p = 25\% \) and 75\% levels; about 6.9\% change is found in the displacement at \( C \), about 29.7\% change is found in the rotation at \( C \), whereas about 7.1\% to 7.3\% changes are noted in the three bending tresses.
Table 5.6 Response values for p-percent probability of success in the fixed beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>$p = 50%$</th>
<th>$p = 25%$</th>
<th>$p = 75%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_1$ (in.-klb)</td>
<td>12334.014</td>
<td>11475.340 (93.04%)</td>
<td>13192.629 (106.96%)</td>
<td></td>
</tr>
<tr>
<td>$M_2$ (in.-klb)</td>
<td>-24652.045</td>
<td>-22965.681 (93.16%)</td>
<td>-26338.409 (106.84%)</td>
<td></td>
</tr>
<tr>
<td>$M_4$ (in.-klb)</td>
<td>13361.896</td>
<td>12499.694 (93.55%)</td>
<td>14224.099 (106.45%)</td>
<td></td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_C$ (in.)</td>
<td>0.653</td>
<td>0.609 (93.26%)</td>
<td>0.698 (106.89%)</td>
<td></td>
</tr>
<tr>
<td>$\theta_C$ (rad)</td>
<td>0.00148</td>
<td>0.00103 (69.59%)</td>
<td>0.00192 (129.73%)</td>
<td></td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_A$ (ksi)</td>
<td>85.677</td>
<td>79.542 (92.84%)</td>
<td>91.813 (107.16%)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_C$ (ksi)</td>
<td>-171.248</td>
<td>-159.157 (92.94%)</td>
<td>-183.338 (107.06%)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_B$ (ksi)</td>
<td>93.077</td>
<td>86.312 (92.73%)</td>
<td>99.843 (107.27%)</td>
<td></td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of moment at $A$, displacement at $C$ and stress at $B$, with respect to the ransom variables at 75% probability of occurrence, is shown in Fig. 5.24 (a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $A$ is sensitive to the distributed load of intensity $w$ and two moments of inertia $I_1$ and $I_2$, the Young’s modulus $E$ has little effect on the moment at $A$. The sensitivity of the displacement $x_C$ is sensitive to the distributed load of intensity $w$, Young’s modulus $E$ and moments of inertia $I_1$ and $I_2$, not sensitive to the depths $d_1$ and $d_2$, coefficient of thermal expansion $\alpha$ and temperatures $T_u$ and $T_l$. In the stress sensitivity, the bending stress at $B$ is sensitive to the depth $d_2$, moment of inertia $I_2$, and the distributed load of intensity $w$, not sensitive to the depth $d_1$ and temperatures $T_u$ and $T_l$.

For stochastic design optimization, it is assumed that the material density has a mean of 0.289 $lbf/in.^3$ and standard deviation of 0.005 $lbf/in.^3$. The allowable strength has
(a) Sensitivity for moment at A.

(b) Sensitivity for the displacement at C.

(c) Sensitivity for the bending stress at B.

Fig. 5.24 Sensitivity analysis of response in the fixed beam.
a mean of 100 ksi and standard deviation of 10.0 ksi. The allowable displacement has a mean of 0.65 in. and standard deviation of 0.065 in. And 0.01 in. ≤ d₁, d₂ ≤ 30.0 in. The optimal results of the beam are shown in Fig. 5.25. The optimization results for both the first- and second-order approximations in the stochastic analysis have little differences, since the variances are small.

(a) Optimal weight versus the probability of occurrence p.

(b) Design variables versus the probability of occurrence p.

Fig. 5.25 Optimal results of the fixed beam.
Example 7: Clamped beam for a mechanical load, a thermal load and support settling.

A uniform beam of thickness $b$ ($b = 6\text{ in}$) and length $l$ ($l = 2a = 240\text{ in}$) is clamped at both ends ($A$ and $B$) as shown in Fig. 5.26(a). Four sizing design variables contain two moments of inertia $I_1$ and $I_2$, and two depths, $d_1$ and $d_2$ for beam spans (AC and CB). Their stochastic properties are as follows:

\[
\begin{align*}
&\mu_{I_1} = \{864\text{ in}^4\}, \quad \sigma_{I_1} = \{14.965\text{ in}^4\} \quad \rho_{I_1} = \begin{bmatrix}
1.00 & 1.010 & 0 & 0 \\
1.010 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 0.956 \\
0 & 0 & 0.956 & 1.000
\end{bmatrix}, \\
&\mu_{I_2} = \{864\text{ in}^4\}, \quad \sigma_{I_2} = \{17.280\text{ in}^4\} \quad \rho_{I_2} = \begin{bmatrix}
1.00 & 1.010 & 0 & 0 \\
1.010 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 0.956 \\
0 & 0 & 0.956 & 1.000
\end{bmatrix}, \\
&\mu_{d_1} = \{12\text{ in}\}, \quad \sigma_{d_1} = \{0.710\text{ in}\} \quad \rho_{d_1} = \begin{bmatrix}
1.00 & 1.010 & 0 & 0 \\
1.010 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 0.956 \\
0 & 0 & 0.956 & 1.000
\end{bmatrix}, \\
&\mu_{d_2} = \{12\text{ in}\}, \quad \sigma_{d_2} = \{0.849\text{ in}\} \quad \rho_{d_2} = \begin{bmatrix}
1.00 & 1.010 & 0 & 0 \\
1.010 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 0.956 \\
0 & 0 & 0.956 & 1.000
\end{bmatrix}.
\end{align*}
\]

The stochastic properties of two material variables, Young’s modulus $E$ and coefficient of thermal expansion $\alpha$ have the following mean value, standard deviation and correlation coefficient matrix:

\[
\begin{align*}
&\mu_E = \{10,000\text{ ksi}\}, \quad \sigma_E = \{670.820\text{ ksi}\} \quad \rho_{E} = \begin{bmatrix}
1.00 & 0.925 \\
0.925 & 1.000
\end{bmatrix}, \\
&\mu_\alpha = \{1.2 \times 10^{-5} \text{ F}^{-1}\}, \quad \sigma_\alpha = \{9.675 \times 10^{-7} \text{ F}^{-1}\} \quad \rho_{\alpha} = \begin{bmatrix}
1.00 & -0.969 \\
-0.969 & 1.000
\end{bmatrix}.
\end{align*}
\]

The beam is subjected to a concentrated load of magnitude $P$ with a mean of 10.0 kip and standard deviation of 0.224 kip. Uniform temperature along the length of the beam, the temperature variation along the depth is linear with values $T_u$ and $T_l$ at the upper and lower surface, respectively, as shown in Fig. 5.26(a). Their mean, standard deviation and correlation coefficient matrix of temperatures are as follows

\[
\begin{align*}
&\mu_{T_u} = \{-10\} \text{ F}, \quad \sigma_{T_u} = \{0.775\} \text{ F} \quad \rho_{T_u} = \begin{bmatrix}
1.00 & -0.969 \\
-0.969 & 1.000
\end{bmatrix}, \\
&\mu_{T_l} = \{10\} \text{ F} \quad \sigma_{T_l} = \{0.866\} \text{ F} \quad \rho_{T_l} = \begin{bmatrix}
1.00 & -0.969 \\
-0.969 & 1.000
\end{bmatrix}.
\end{align*}
\]

The settling of supports A and B by $\Delta_A$ and $\Delta_B$ has the following stochastic properties:

\[
\begin{align*}
&\mu_{\Delta_A} = \{-0.1\} \text{ in}, \quad \sigma_{\Delta_A} = \{0.00632\} \text{ in} \quad \rho_{\Delta_A} = \begin{bmatrix}
1.00 & 0.589 \\
0.589 & 1.000
\end{bmatrix}, \\
&\mu_{\Delta_B} = \{-0.05\} \text{ in}, \quad \sigma_{\Delta_B} = \{0.00335\} \text{ in} \quad \rho_{\Delta_B} = \begin{bmatrix}
1.00 & 0.589 \\
0.589 & 1.000
\end{bmatrix}.
\end{align*}
\]
It is obvious that the problem has eleven random variables defined as follows:

\[
I_1 = \mu_{I_1} (1 + q_{I_1}) = \mu_I (1 + q_I) \quad P = \mu_P (1 + q_P) = \mu_P (1 + q_P)
\]

\[
I_2 = \mu_{I_2} (1 + q_{I_2}) = \mu_2 (1 + q_2) \quad T_u = \mu_{T_u} (1 + q_{T_u}) = \mu_a (1 + q_a)
\]

\[
d_1 = \mu_{d_1} (1 + q_{d_1}) = \mu_3 (1 + q_3) \quad T_i = \mu_{T_i} (1 + q_{T_i}) = \mu_y (1 + q_y)
\]

\[
d_2 = \mu_{d_2} (1 + q_{d_2}) = \mu_4 (1 + q_4) \quad \Delta_A = \mu_\Delta (1 + q_\Delta) = \mu_\Delta (1 + q_\Delta)
\]

\[
E = \mu_E (1 + q_E) = \mu_e (1 + q_e) \quad \Delta_B = \mu_\Delta (1 + q_\Delta) = \mu_\Delta (1 + q_\Delta)
\]

\[
\alpha = \mu_\alpha (1 + q_\alpha) = \mu_\alpha (1 + q_\alpha)
\]

Thus, the stochastic responses can be obtained as follows:

\[
\text{Moment: } \left\{ \begin{array}{c}
\mu_{\sigma_I} \\
\mu_{\sigma_2} \\
\mu_{\sigma_4} \\
\mu_{\sigma_4}
\end{array} \right\}^T \left\{ \begin{array}{c}
\mu_{M_1} \\
\mu_{M_2} \\
\mu_{M_4} \\
\mu_{M_4}
\end{array} \right\} = \left\{ \begin{array}{c}
600.600 \\
45.600 \\
690.600
\end{array} \right\} \text{ in. - klb}
\]

\[
\left\{ \begin{array}{c}
\mu_{M_1} \\
\mu_{M_2} \\
\mu_{M_4} \\
\mu_{M_4}
\end{array} \right\} = \left\{ \begin{array}{c}
603.412 \\
48.797 \\
694.182
\end{array} \right\} \text{ in. - klb .}
\]

Fig. 5.26 Clamped beam under a concentrated load, thermal load and settling support.
\[
\begin{bmatrix}
\sigma_{M_1} \\
\sigma_{M_2} \\
\sigma_{M_3} \\
\sigma_{M_4}
\end{bmatrix} = \begin{bmatrix}
59.104 \\
62.285 \\
62.285 \\
66.317
\end{bmatrix} \text{in.-lb}
\]
\[
\begin{bmatrix}
\rho_M
\end{bmatrix} = \begin{bmatrix}
1.000 & 0.968 & 0.968 & 0.973 \\
0.968 & 1.000 & 1.000 & 0.972 \\
0.968 & 1.000 & 1.000 & 0.972 \\
0.973 & 0.972 & 0.972 & 1.000
\end{bmatrix}.
\]

Displacement/rotation:
\[
\begin{bmatrix}
\mu_{\gamma_c} \\
\mu_{\theta_c}
\end{bmatrix} = \begin{bmatrix}
\mu_{\gamma_c} \\
\mu_{\theta_c}
\end{bmatrix}^T = \begin{bmatrix}
0.158\text{in.} \\
-0.3125 \times 10^{-3}\text{rad}
\end{bmatrix} \quad \begin{bmatrix}
\mu_{\gamma_c} \\
\mu_{\theta_c}
\end{bmatrix}^T = \begin{bmatrix}
0.159\text{in.} \\
-0.3116 \times 10^{-3}\text{rad}
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\sigma_{\gamma_c} \\
\sigma_{\theta_c}
\end{bmatrix} = \begin{bmatrix}
0.00750\text{in.} \\
0.0000347\text{rad}
\end{bmatrix}
\]

Stress:
\[
\begin{bmatrix}
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{bmatrix} = \begin{bmatrix}
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{bmatrix}^T = \begin{bmatrix}
4.171 \\
0.317 \\
0.317 \\
4.796
\end{bmatrix} \text{ksi}
\]
\[
\begin{bmatrix}
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{bmatrix}^T = \begin{bmatrix}
4.183 \\
0.329 \\
0.327 \\
4.809
\end{bmatrix} \text{ksi}.
\]

\[
\begin{bmatrix}
\sigma_{\sigma_1} \\
\sigma_{\sigma_2} \\
\sigma_{\sigma_3} \\
\sigma_{\sigma_4}
\end{bmatrix} = \begin{bmatrix}
0.403 \\
0.426 \\
0.425 \\
0.452
\end{bmatrix} \text{ksi}
\]
\[
\begin{bmatrix}
\rho_M
\end{bmatrix} = \begin{bmatrix}
1.000 & 0.777 & 0.782 & 0.983 \\
0.777 & 1.000 & 1.000 & 0.731 \\
0.782 & 1.000 & 1.000 & 0.736 \\
0.983 & 0.731 & 0.736 & 1.000
\end{bmatrix}.
\]

Note that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of three moments are about 9.8% at $A$, 136.6% at $C$ and 9.6% at $B$, respectively; the standard deviation of the displacement is 4.7% at $C$, the standard deviation of rotation are 11.1% at $C$; the standard deviations of bending stresses are about 9.7%, 134.4% and 9.4%, respectively.

The probability density function and cumulative distribution function for the bending stress at $C$ are shown in Fig. 5.27(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description have little differences.
Fig. 5.27 Stochastic description of the bending stress at C.

The calculated response values for different probability levels of occurrences \( p = 50\%, 25\% \) and \( 75\% \) are listed in Table 5.7. In the table, it may be noted that about 6.6\%, 86.1\% and 6.4\% percent changes are found in the three moments in the \( p = 25\% \) and \( 75\% \) levels; about 3.1\% change is found in the displacement at C, about 7.4\% change is found in the rotation at C, whereas about 6.5\%, 87.2\% and 6.3\% changes are noted in the three bending tresses.
Table 5.7 Response values for p-percent probability of success in the clamped beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Moment</td>
<td></td>
</tr>
<tr>
<td>$M_1$ (in.-klb)</td>
<td>603.412</td>
</tr>
<tr>
<td>$M_2/M_3$ (in.-klb)</td>
<td>48.797</td>
</tr>
<tr>
<td>$M_4$ (in.-klb)</td>
<td>694.182</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
</tr>
<tr>
<td>$v_C$ (in.)</td>
<td>0.159</td>
</tr>
<tr>
<td>$\theta_B$ (rad)</td>
<td>-0.000312</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_A$ (ksi)</td>
<td>4.183</td>
</tr>
<tr>
<td>$\sigma_C$ (ksi)</td>
<td>0.329</td>
</tr>
<tr>
<td>$\sigma_B$ (ksi)</td>
<td>4.809</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of moment at $C$, displacement at $C$ and bending stress at $A$, with respect to the random variables at 75\% probability of occurrence, is shown in Fig. 5.28(a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $C$ is the most sensitive to the Young’s modulus $E$, coefficient of thermal expansion $\alpha$ and load $P$, not sensitive to the settling supports $\Delta_A$ and $\Delta_B$. The sensitivity of the displacement $x_C$ is the most sensitive to the Young’s modulus $E$ and load $P$, not sensitive to the depths $d_1$ and $d_2$, coefficient of thermal expansion $\alpha$ and temperatures $T_u$ and $T_i$. In the stress sensitivity, the bending stress at $A$ is the most sensitive to the coefficient of thermal expansion $\alpha$.  

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Fig. 5.28 Sensitivity analysis of response in the clamped beam.
In stochastic design optimization, it is assumed that the material density has a mean of 0.289 \( \text{lbf/in.}^3 \) and standard deviation of 0.005 \( \text{lbf/in.}^3 \). The allowable strength has a mean of 3800 \( \text{psi} \) and standard deviation of 380 \( \text{psi} \). The allowable displacement has a mean of 0.28 \( \text{in.} \) and standard deviation of 0.028 \( \text{in.} \). And \( 0.001 \text{in.} \leq d_1, d_2 \leq 80.0 \text{in.} \). The optimal results are shown in Fig. 5.29. The optimization results for both the first- and second-order approximations in the stochastic analysis are almost equal, since the variances are small.

(a) Optimal weight versus the probability of occurrence \( p \).

(b) Design variables versus the probability of occurrence \( p \).

Fig. 5.29 Optimal results of the clamped beam.
Example 8: Torsion of a shaft fixed at both ends.

A circular shaft shown in Fig. 5.30(a) is fixed at both ends and made of two materials with shear modulus $G_1$ and $G_2$. Their mean, standard deviation and correlation coefficient matrix as follows:

\[
\begin{align*}
\{\mu_{G_1}\} &= \{6,500ksi\} \\
\{\mu_{G_2}\} &= \{4,000ksi\} \\
\{\sigma_{G_1}\} &= \{290.689ksi\} \\
\{\sigma_{G_2}\} &= \{178.885ksi\} \\
[C_{\text{mv}}] &= \begin{bmatrix} 1.000 & 0.500 \\ 0.500 & 1.000 \end{bmatrix}.
\end{align*}
\]

![Composite shaft.](image)

(a) Composite shaft.

![Free body diagram.](image)

(b) Free body diagram.

Fig.5.30 Torsion of a circular shaft fixed at both ends.

The sizing design variables consist of two polar moments of inertia, $J_1$ and $J_2$, and two diameters, $r_1$ and $r_2$. Their stochastic properties are shown as follows:

\[
\begin{align*}
\{\mu_{J_1}\} &= \{25.133in.\}^4 \\
\{\mu_{J_2}\} &= \{23.562in.\}^4 \\
\{\mu_{r_1}\} &= \{2.0in.\} \\
\{\mu_{r_2}\} &= \{1.0in.\} \\
\{\sigma_{J_1}\} &= \{1.257in.\}^4 \\
\{\sigma_{J_2}\} &= \{0.942in.\}^4 \\
\{\sigma_{r_1}\} &= \{0.028in.\} \\
\{\sigma_{r_2}\} &= \{0.02in.\} \\
[C_{\text{dv}}] &= \begin{bmatrix} 1.000 & 0.0625 & 0 & 0 \\ 0.0625 & 1.000 & 0 & 0 \\ 0 & 0 & 1.000 & 0.566 \\ 0 & 0 & 0.566 & 1.000 \end{bmatrix}.
\end{align*}
\]
The shaft, which has a total length of \( l (l = 138 \text{in.}) \), is subjected to a torque \( \bar{T} \) at a distance \( a = 78 \text{in.} \) from its left support. The torque \( \bar{T} \) has a mean of 40,000 \( \text{in.-lb} \) and standard deviation of 400 \( \text{in.-lb} \). The support settles at ends \( A \) and \( B \) with stochastic properties:

\[
\begin{align*}
\begin{bmatrix} 
\mu_{\phi_1} \\
\mu_{\phi_2}
\end{bmatrix} &= 
\begin{bmatrix} 
0.002 \\
-0.004
\end{bmatrix} \text{rad} \\
\begin{bmatrix} 
\sigma_{\phi_1} \\
\sigma_{\phi_2}
\end{bmatrix} &= 
\begin{bmatrix} 
4.0 \times 10^{-3} \\
8.0 \times 10^{-3}
\end{bmatrix} \text{rad} \\
\rho_{\phi} &= 
\begin{bmatrix} 
1.000 \\
-0.400 \\
-0.400 \\
1.000
\end{bmatrix}.
\end{align*}
\]

We note that the problem has nine random variables defined as:

\[
\begin{align*}
J_1 &= \mu_{J_1} \left( 1 + q_{J_1} \right) = \mu_J \left( 1 + q_J \right) \quad G_1 &= \mu_{G_1} \left( 1 + q_{G_1} \right) = \mu_J \left( 1 + q_J \right) \\
J_2 &= \mu_{J_2} \left( 1 + q_{J_2} \right) = \mu_J \left( 1 + q_J \right) \quad G_2 &= \mu_{G_2} \left( 1 + q_{G_2} \right) = \mu_J \left( 1 + q_J \right) \\
r_1 &= \mu_{r_1} \left( 1 + q_{r_1} \right) = \mu_J \left( 1 + q_J \right) \quad T &= \mu_T \left( 1 + q_T \right) = \mu_J \left( 1 + q_J \right) \\
r_2 &= \mu_{r_2} \left( 1 + q_{r_2} \right) = \mu_J \left( 1 + q_J \right) \quad \varphi_1 &= \mu_{\varphi_1} \left( 1 + q_{\varphi_1} \right) = \mu_J \left( 1 + q_J \right) \\
& \quad \varphi_2 = \mu_{\varphi_2} \left( 1 + q_{\varphi_2} \right) = \mu_J \left( 1 + q_J \right)
\end{align*}
\]

Thus, the stochastic responses can be obtained as follows:

**Toque:**

\[
\begin{align*}
\begin{bmatrix} 
\mu_{T_a} \\
\mu_{T_c}
\end{bmatrix} &= 
\begin{bmatrix} 
24.652 \\
-15.348
\end{bmatrix} \text{in.-k lb} \\
\begin{bmatrix} 
\sigma_{T_a} \\
\sigma_{T_c}
\end{bmatrix} &= 
\begin{bmatrix} 
2.408 \\
1.875
\end{bmatrix} \text{in.-k lb} \\
\rho_T &= 
\begin{bmatrix} 
1.000 \\
0.741 \\
-0.741 \\
1.000
\end{bmatrix}.
\end{align*}
\]

**Twist angle:**

\[
\mu_\varphi = \mu_\varphi = 0.01377 \text{rad} \quad \mu_\varphi = 0.01380 \text{rad} \quad \sigma_\varphi = 0.001234 \text{rad}
\]

**Stress:**

\[
\begin{align*}
\begin{bmatrix} 
\mu_{\sigma_a} \\
\mu_{\sigma_c}
\end{bmatrix} &= 
\begin{bmatrix} 
1.9617 \\
-0.6514
\end{bmatrix} \text{ksi} \\
\begin{bmatrix} 
\sigma_{\sigma_a} \\
\sigma_{\sigma_c}
\end{bmatrix} &= 
\begin{bmatrix} 
0.1980 \\
0.0805
\end{bmatrix} \text{ksi} \\
\mu_\sigma &= 
\begin{bmatrix} 
1.9638 \\
-0.6523
\end{bmatrix} \text{ksi} \\
\sigma_\sigma &= 
\begin{bmatrix} 
1.0000 \\
0.9051 \\
-0.9051 \\
1.0000
\end{bmatrix}.
\end{align*}
\]

Note that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of two toques are about 9.8% at \( A \) and 12.2% at \( C \),
respectively; the standard deviation of the twist angle is 9.0% at $B$; the standard deviations of bending stresses are about 10.1% and 12.4%, respectively.

The probability density function and cumulative distribution function for torque at $A$ and torque load $\bar{T}$ are shown in Fig. 5.31. It is obvious that both the first- and second-order approximations in the probabilistic description are almost equal to each other.

![Graph showing probability density functions for the internal torque and torque load.](image)

(a) Probability density functions for the internal torque and torque load.

![Graph showing cumulative distribution functions for the internal torque and torque load.](image)

(b) Cumulative distribution functions for the internal torque and torque load.

Fig. 5.31 Stochastic description of the torques.
The calculated response variables for different probability levels of occurrence (\( p = 50\%, 25\% \) and 75\%) are listed in Tables 5.8. Noted in the table that about 6.6\% and 8.2\% percent changes are found in the two torque in the \( p = 25\% \) and 75\% levels; about 6.0\% change is found in the twist angle at \( C \), whereas about 6.8\% and 8.4\% changes are noted in the two tresses.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>( p = 50% )</th>
<th>( p = 25% )</th>
<th>( p = 75% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torque ( T_A ) (in.-klb)</td>
<td>24.641</td>
<td>23.017 (93.41%)</td>
<td>26.266 (106.59%)</td>
<td></td>
</tr>
<tr>
<td>Torque ( T_C ) (in.-klb)</td>
<td>-15.359</td>
<td>-14.094 (91.76%)</td>
<td>-16.623 (108.23%)</td>
<td></td>
</tr>
<tr>
<td>Twist Angle ( \theta_B ) (rad)</td>
<td>0.01380</td>
<td>0.01297 (93.99%)</td>
<td>0.01463 (106.01%)</td>
<td></td>
</tr>
<tr>
<td>Shear Stress ( \sigma_A ) (ksi)</td>
<td>1.964</td>
<td>1.830 (93.18%)</td>
<td>2.097 (106.77%)</td>
<td></td>
</tr>
<tr>
<td>Shear Stress ( \sigma_C ) (ksi)</td>
<td>-0.652</td>
<td>-0.598 (91.72%)</td>
<td>-0.707 (108.44%)</td>
<td></td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of torque at \( A \), twist angle at \( B \) and shear stress at \( B \), with respect to the random variables at the 75\% probability of occurrence, is shown in Fig. 5.32(a), (b) and (c), respectively. It is obvious that there is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other. In the torque sensitivity, the moment at \( A \) is the most sensitive to the external torque \( T \), not sensitive to two diameters, \( r_1 \) and \( r_2 \). The sensitivity of the twist angle \( \varphi \) is the most sensitive to the external torque \( T \), not sensitive to two diameters, \( r_1 \) and \( r_2 \). In the stress sensitivity, the shear stress at \( B \) is the most sensitive to the external torque \( T \) and the diameter \( r_2 \).
(a) Sensitivity for the internal torque.

(b) Sensitivity for the twist angle.

(c) Sensitivity for the shear stress at $B$.

Fig. 5.32 Sensitivity analysis of response in the shaft.
In stochastic design optimization, it is assumed that the stochastic properties of the material densities are as follows:

\[
\begin{pmatrix}
\mu_{\rho_1} \\
\mu_{\rho_2}
\end{pmatrix} = \begin{pmatrix}
0.1 \\
0.3
\end{pmatrix} \text{lb} / \text{in.}^3 \\
\begin{pmatrix}
\sigma_{\rho_1} \\
\sigma_{\rho_2}
\end{pmatrix} = \begin{pmatrix}
0.005 \\
0.03
\end{pmatrix} \text{lb} / \text{in.}^3 \\
[\rho_\rho] = \begin{pmatrix}
1.00 & 0.50 \\
0.50 & 1.00
\end{pmatrix}.
\]

The allowable strength of the first material has a mean of 2.0 ksi and standard deviation of 0.2 ksi, and the allowable strength of the second material has a mean of 0.5 ksi and standard deviation of 0.05 ksi. The allowable twist angle has a mean of 0.012 rad and standard deviation of 0.0012 rad. And \(0.01 \text{ in.} \leq d_1, d_2 \leq 0.80 \text{ in.}\) As a result, the stochastic optimal results are shown in Fig. 5.33. The optimization results for both the first- and second-order approximations in the stochastic analysis are almost equal, since the variances are small.

(a) Optimal weight versus the probability of occurrence \(p\).

Fig. 5.33 Optimal results of the fixed shaft.
Example 9: Beam supported by a tie rod.

A steel beam of length $L$ ($L = 118.11\text{ in.}$) modulus of elasticity $E_s$ ($\mu_{E_s} = 29,007.55\text{ ksi}$ and $\sigma_{E_s} = 1,834.599\text{ ksi}$) is fixed at C and supported by a tie rod at B as shown in Fig. 5.34. The sizing design variables consist of two moments of inertia $I_1$ and $I_2$, and two depths, $d_1$ and $d_2$ for beam spans (BD and DC). Their stochastic properties are as follows:

\[
\begin{align*}
\mu_{I_1} &= 48.050\text{ in.}^4 \\
\mu_{I_2} &= 48.050\text{ in.}^4 \\
\mu_{d_1} &= 7.874\text{ in.} \\
\mu_{d_2} &= 7.874\text{ in.} \\
\sigma_{I_1} &= 8.989\text{ in.}^4 \\
\sigma_{I_2} &= 10.193\text{ in.}^4 \\
\sigma_{d_1} &= 1.409\text{ in.} \\
\sigma_{d_2} &= 1.575\text{ in.} \\
\end{align*}
\]

\[
\begin{bmatrix}
\mu_{\Delta I_1} \\
\mu_{\Delta I_2} \\
\mu_{\Delta d_1} \\
\mu_{\Delta d_2}
\end{bmatrix} =
\begin{bmatrix}
1.000 & 0.630 & 0 & 0 \\
0.630 & 1.000 & 0 & 0 \\
0 & 0 & 1.000 & 1.006 \\
0 & 0 & 1.006 & 1.000
\end{bmatrix}
\]

The beam is subjected to a uniformly distributed load of intensity $w$ per unit length ($\mu_w = 68.522\text{ lb/in.}$ and $\sigma_w = 6.852\text{ lb/in.}$). The tie rod, which is made of aluminum, has a cross-sectional area of $A_t$ ($\mu_{A_t} = 0.155\text{ in.}^2$ and $\sigma_{A_t} = 0.031\text{ in.}^2$), a
modulus of elasticity of $E_i$ ($\mu_{E_i} = 10,152.64 ksi$ and $\sigma_{E_i} = 717.90 ksi$) and length $l$ ($l = 295.28\,\text{in.}$).

It is obvious that the problem has eight random variables defined as follows:

\begin{align*}
A_t &= \mu_{A_t} \left(1 + q_{A_t}\right) = \mu_t \left(1 + q_1\right) & d_2 &= \mu_{d_2} \left(1 + q_{d_2}\right) = \mu_5 \left(1 + q_5\right) \\
I_1 &= \mu_{I_1} \left(1 + q_{I_1}\right) = \mu_2 \left(1 + q_2\right) & E_t &= \mu_{E_t} \left(1 + q_{E_t}\right) = \mu_6 \left(1 + q_6\right) \\
I_2 &= \mu_{I_2} \left(1 + q_{I_2}\right) = \mu_3 \left(1 + q_3\right) & E_b &= \mu_{E_b} \left(1 + q_{E_b}\right) = \mu_7 \left(1 + q_7\right)
\end{align*}
Thus, the selected stochastic responses can be obtained as follows:

Thus, the selected stochastic responses can be obtained as follows:

\[
\begin{bmatrix}
\mu_F \\
\mu_{\bar{\pi}_2} \\
\mu_{\bar{\pi}_4} \\
\mu_{\pi} \\
\mu_{\nu_D} \\
\mu_{\theta_D} \\
\mu_{\sigma_D} \\
\mu_{\sigma_C}
\end{bmatrix} =
\begin{bmatrix}
\mu_{\bar{F}} \\
\mu_{\bar{\pi}_2} \\
\mu_{\bar{\pi}_4} \\
\mu_{\pi} \\
\mu_{\nu_D} \\
\mu_{\theta_D} \\
\mu_{\sigma_D} \\
\mu_{\sigma_C}
\end{bmatrix}^I
\begin{bmatrix}
2.056\text{klb} \\
121.412\text{in.} - \text{klb} \\
-235.116\text{in.} - \text{klb} \\
0.386\text{in.} \\
-0.170\text{in.} \\
0.00410\text{rad} \\
9.948\text{ksi} \\
-19.264\text{ksi}
\end{bmatrix}
\begin{bmatrix}
\mu_F \\
\mu_{\bar{\pi}_2} \\
\mu_{\bar{\pi}_4} \\
\mu_{\pi} \\
\mu_{\nu_D} \\
\mu_{\theta_D} \\
\mu_{\sigma_D} \\
\mu_{\sigma_C}
\end{bmatrix}^I
\begin{bmatrix}
2.047\text{klb} \\
120.901\text{in.} - \text{klb} \\
-236.138\text{in.} - \text{klb} \\
0.396\text{in.} \\
-0.176\text{in.} \\
0.00420\text{rad} \\
10.332\text{ksi} \\
-19.917\text{ksi}
\end{bmatrix}
\]

Note that there is a small difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of force is about 14.2% at rod, the standard deviations of two moments are about 14.2% at D and 14.4% at C, respectively; the standard deviations of the displacements are 19.0% at rod and 18.2%, the standard deviation of rotation are 18.3% at D; the standard deviations of bending stresses are about 32.1% and 27.4%, respectively.

The probability density function and cumulative distribution function for the moment at D are shown in Fig. 5.35(a) and (b), respectively. It is obvious that both the
first- and second-order approximations in the probabilistic description have small differences.

The evaluated responses for different probabilities levels of occurrence ($p = 50\%, 25\%$ and $75\%$) are listed in Table 5.9. In the table, it may be noted that about 9.6%
percent changes are found in the force and two moments in the $p = 25\%$ and $75\%$ levels; about $12.4\%$ and $11.9\%$ changes are found in the displacement at rod and at $D$, about $12.1\%$ change is found in the rotation at $D$, whereas about $20.8\%$ and $17.9\%$ changes are noted in the two bending tresses.

Table 5.9 Response values for $p$-percent probability of success in the tied beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Moment</td>
<td></td>
</tr>
<tr>
<td>$F$ (k lb)</td>
<td>2.047</td>
</tr>
<tr>
<td>$M_2$ (in.-k lb)</td>
<td>120.901</td>
</tr>
<tr>
<td>$M_4$ (in.-k lb)</td>
<td>-236.138</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
</tr>
<tr>
<td>$x$ (in.)</td>
<td>0.396</td>
</tr>
<tr>
<td>$v_D$ (in.)</td>
<td>-0.176</td>
</tr>
<tr>
<td>$\theta_D$ (rad)</td>
<td>0.00420</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_D$ (ksi)</td>
<td>10.332</td>
</tr>
<tr>
<td>$\sigma_C$ (ksi)</td>
<td>-19.917</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of moment at $C$, displacement at $D$ and stress at the rod, with respected to random variables at $75\%$ probability of occurrence, is shown in Fig. 5.36(a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $C$ is sensitive to the distributed load of intensity $w$, not sensitive to the depths $d_1$ and $d_2$. The sensitivity of the displacement $x_0$ is similar to The sensitivity of the moment at $C$. In the tension stress sensitivity, the tension stress at rod is also sensitive to the distributed load of intensity $w$ and the rod area $A$, not sensitive to the depths $d_1$ and $d_2$. 

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(a) Sensitivity for the moment at $C$.

(b) Sensitivity for the displacement at $D$.

(c) Sensitivity for the tension stress at the rod.

Fig. 5.36 Sensitivity analysis of response in the beam.
In stochastic design optimization, it is assumed that the two material densities have mean values of $\mu_{\rho_1} = 0.124 \text{lbf/in.}^3$ and $\mu_{\rho_2} = 0.289 \text{lbf/in.}^3$ and standard deviations of $\sigma_{\rho_1} = 0.002 \text{lbf/in.}^3$ and $\sigma_{\rho_2} = 0.005 \text{lbf/in.}^3$. The allowable tension strength has a mean of 10.0 $\text{ksi}$ and standard deviation of 1.0 $\text{ksi}$. The allowable bending strength has a mean of 15.0 $\text{ksi}$ and standard deviation of 1.5 $\text{ksi}$. The allowable displacement at rod has a mean of 0.25 $\text{in}$ and standard deviation of 0.025 $\text{in}$. The allowable displacement at $D$ has a mean of 0.15 $\text{in}$ and standard deviation of 0.015 $\text{in}$. $0.01 \text{in.}^2 \leq A \leq 10.0 \text{in.}^2$ and $1.5 \text{in.} \leq d_1, d_2 \leq 36.0 \text{in}$. Therefore, the optimal results for different probability of occurrence $p$ are demonstrated in Fig. 5.37. The optimization results for both the first- and second-order approximations in the stochastic analysis have small differences and the design variables have some changes.

(a) Optimal weight versus the probability of occurrence $p$.

Fig. 5.37 Optimal results of the beam supported by a tie rod.
Example 10: Three-bar truss for mechanical, thermal and settling support loads.

A three-bar truss made of steel has a Young’s modulus $E$ with a mean, $\mu_E$, of 30,000 $ksi$ and standard deviation, $\sigma_E$, of 3,000 $ksi$, and a coefficient of thermal expansion, $\alpha$, with a mean, $\mu_\alpha$, of $6.6 \times 10^{-6}/^\circ F$ and standard deviation, $\sigma_\alpha$, of $6.6 \times 10^{-7}/^\circ F$ as shown in Fig.5.38. The sizing variables, three areas $A_1$, $A_2$ and $A_3$, have the mean, standard deviation and correlation coefficient matrix as follows:
The mean, standard deviation and correlation matrix for the mechanical loads \((x_P, y_P)\) are as follows:

\[
\begin{bmatrix}
\mu_{x_P} \\
\mu_{y_P} \\
\sigma_{x_P} \\
\sigma_{y_P}
\end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 0.1 \\ 0.1 \end{bmatrix} \text{kip}, \quad \begin{bmatrix} \mu_{x_P} \\
\mu_{y_P} \\
\sigma_{x_P} \\
\sigma_{y_P}
\end{bmatrix} = \begin{bmatrix} 5.00 \\ 5.00 \\ 0.005 \\ 0.015 \end{bmatrix} \text{kip}, \quad \rho = \begin{bmatrix} 1.00 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.25 \\ 0.25 & 0.25 & 1.00 \end{bmatrix}.
\]

Support node 2 settles in both \(x\) and \(y\) directions \(\Delta_x\) and \(\Delta_y\) with the following stochastic properties:

\[
\begin{bmatrix}
\mu_{\Delta_x} \\
\mu_{\Delta_y} \\
\sigma_{\Delta_x} \\
\sigma_{\Delta_y}
\end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.005 \\ 0.015 \end{bmatrix} \text{in.}, \quad \begin{bmatrix} \mu_{\Delta_x} \\
\mu_{\Delta_y} \\
\sigma_{\Delta_x} \\
\sigma_{\Delta_y}
\end{bmatrix} = \begin{bmatrix} 1.00 \\ 0.25 \times 10^{-2} \\ 0.25 \times 10^{-2} \\ 1.00 \end{bmatrix}.
\]

The temperature design, \(T\), at node 1 has a mean, \(\mu_T\), of 100°F and a standard deviation, \(\sigma_T\), of 10°F.
We note that the problem has ten random variables defined as follows:

\[ A_1 = \mu_A (1 + q_A) = \mu_1 (1 + q_1) \quad P_x = \mu_P (1 + q_P) = \mu_6 (1 + q_6) \]
\[ A_2 = \mu_A (1 + q_A) = \mu_2 (1 + q_2) \quad P_y = \mu_P (1 + q_P) = \mu_7 (1 + q_7) \]
\[ A_3 = \mu_A (1 + q_A) = \mu_3 (1 + q_3) \quad T = \mu_T (1 + q_T) = \mu_8 (1 + q_8) \]
\[ E = \mu_E (1 + q_E) = \mu_4 (1 + q_4) \quad \Delta_{2x} = \mu_{\Delta_x} (1 + q_{\Delta_x}) = \mu_9 (1 + q_9) \]
\[ \alpha = \mu_\alpha (1 + q_\alpha) = \mu_5 (1 + q_5) \quad \Delta_{2y} = \mu_{\Delta_y} (1 + q_{\Delta_y}) = \mu_{10} (1 + q_{10}) \]

Thus, the stochastic responses can be obtained as follows:

**Force:**

\[
\begin{bmatrix}
\mu_{\vec{F}_1} \\
\mu_{\vec{F}_2} \\
\mu_{\vec{F}_3}
\end{bmatrix}
= 
\begin{bmatrix}
\mu_{\vec{F}_1} \\
\mu_{\vec{F}_2} \\
\mu_{\vec{F}_3}
\end{bmatrix}
\begin{bmatrix}
62.780 \\
61.215 \\
-7.930
\end{bmatrix}
\text{kip} \quad \begin{bmatrix}
\mu_{\vec{F}_1} \\
\mu_{\vec{F}_2} \\
\mu_{\vec{F}_3}
\end{bmatrix}
= 
\begin{bmatrix}
62.765 \\
61.237 \\
-7.946
\end{bmatrix}
\text{kip}.
\]

**Displacement:**

\[
\begin{bmatrix}
\sigma_{\vec{x}_1} \\
\sigma_{\vec{x}_2} \\
\sigma_{\vec{x}_3}
\end{bmatrix}
= 
\begin{bmatrix}
0.197 \\
-0.237 \\
-0.237
\end{bmatrix}
\text{in.} \quad \begin{bmatrix}
\sigma_{\vec{y}_1} \\
\sigma_{\vec{y}_2} \\
\sigma_{\vec{y}_3}
\end{bmatrix}
= 
\begin{bmatrix}
0.201 \\
-0.241 \\
-0.241
\end{bmatrix}
\text{in.}
\]

**Stress:**

\[
\begin{bmatrix}
\sigma_{\vec{\sigma}_1} \\
\sigma_{\vec{\sigma}_2} \\
\sigma_{\vec{\sigma}_3}
\end{bmatrix}
= 
\begin{bmatrix}
62.780 \\
61.215 \\
-3.965
\end{bmatrix}
\text{ksi} \quad \begin{bmatrix}
\sigma_{\vec{\sigma}_1} \\
\sigma_{\vec{\sigma}_2} \\
\sigma_{\vec{\sigma}_3}
\end{bmatrix}
= 
\begin{bmatrix}
63.284 \\
61.702 \\
-3.982
\end{bmatrix}
\text{ksi}.
\]

In the stochastic responses, it can be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of the bar forces are about 7% for the first two bars and 59% for the third bar; the standard deviation of the
displacements are 18% and 12%; the standard deviation of the stresses are about 10% for the first two bars and 59% for the third bar.

The probability density function and cumulative distribution function for displacements, $x_1$ and $x_2$, are shown in Fig. 5.39(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description have small differences.

(a) Probability density functions for displacements, $x_1$ and $x_2$.

(b) Cumulative distribution functions for displacements, $x_1$ and $x_2$. 

Fig. 5.39 Stochastic description of the displacements, $x_1$ and $x_2$. 

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The evaluated responses for different probability levels of occurrence ($p = 50\%$, $25\%$ and $75\%$) are listed in Table 5.10. In the table, about five percent change is found in the first and second bar force in the $p = 25\%$ and $75\%$ levels; about 40\% change is noted in the third bar force; about 12\% and 8\% changes are found in the horizontal and vertical displacements, respectively; about 7.2\%, 6.8\% and 39.6\% changes are found in three stresses.

Table 5.10 Response values for p-percent probability of success in the three-bar truss.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Column Force</td>
<td></td>
</tr>
<tr>
<td>$F_1$ (kip)</td>
<td>62.765</td>
</tr>
<tr>
<td>$F_2$ (kip)</td>
<td>61.237</td>
</tr>
<tr>
<td>$F_3$ (kip)</td>
<td>-7.946</td>
</tr>
<tr>
<td>Displacements</td>
<td></td>
</tr>
<tr>
<td>$x$ (in.)</td>
<td>0.201</td>
</tr>
<tr>
<td>$y$ (in.)</td>
<td>-0.241</td>
</tr>
<tr>
<td>Column Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_1$ (ksi)</td>
<td>63.284</td>
</tr>
<tr>
<td>$\sigma_2$ (ksi)</td>
<td>61.702</td>
</tr>
<tr>
<td>$\sigma_3$ (ksi)</td>
<td>-3.982</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis at the 75\% probability of occurrence with respect to the random variables, is given for the second bar force, the horizontal displacement and stress in third bar in Fig.5.40(a), (b) and (c), respectively. Note that there is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are different. In the force sensitivity, the second bar force is sensitive to the load $P_y$, bar areas $A_1$ and $A_2$, not sensitive to the Young’s modulus $E$. In the displacement sensitivity, the horizontal displacement is
Fig. 5.40 Sensitivity analysis of responses in the three-bar truss.

(a) Sensitivity for the second bar force.

(b) Sensitivity for the horizontal displacement.

(c) Sensitivity for stress in the third bar.
sensitive to the Young’s modulus $E$, the first bar area $A_1$, and load $P_x$, not sensitive to temperature $T$ and coefficient of thermal expansion $\alpha$. In the stress sensitivity, the third bar stress is sensitive to the loads $P_x$ and $P_y$ and bar areas $A_1$ and $A_2$, not sensitive to the Young’s modulus $E$ (see Appendix 2. and 3.).

In stochastic design optimization, it is assumed that the material density has a mean of 0.289 $lbf/in.^3$ and standard deviation of 0.005 $lbf/in.^3$. The allowable strength has a mean of 20.0 $ksi$ and standard deviation of 2.0 $ksi$. The allowable displacement has a mean of 0.2 in. and standard deviation of 0.015 in. $0.05in.^2 \leq A_1, A_2, A_3 \leq 10.0in.^2$. The optimal results for three-bar truss are shown in Fig. 5.41. The optimization results for both the first- and second-order approximations in the stochastic analysis have little differences, since the variances are small.

(a) Optimal weight versus the probability of occurrence $p$.

Fig. 5.41 Optimal results of the three-bar truss.
Example 11: Six-bar truss for mechanical and thermal loads.

The six-bar truss shown in Fig. 5.42 has a Young’s modulus $E$ with a mean of 10,000 ksi and standard deviation of 173.205 ksi, and a coefficient of thermal expansion $\alpha$ with a mean of $6.0 \times 10^{-6}/^\circ F$ and standard deviation of $2.324 \times 10^{-7}/^\circ F$. It is subjected to a mechanical load $P$ with a mean of 1.0 kip and standard deviation of 0.0158 kip at node 1 along the $y$-direction. The temperature on member 3 has an increasing mean of 100$^\circ F$ and standard deviation of 2.550$^\circ F$. The sizing variables, six cross-sectional areas, have the mean, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{A_1} \\
\mu_{A_2} \\
\mu_{A_3} \\
\mu_{A_4} \\
\mu_{A_5} \\
\mu_{A_6}
\end{bmatrix} = 
\begin{bmatrix}
1.000 \\
0.707 \\
1.000 \\
0.707 \\
1.000 \\
1.000
\end{bmatrix}
\quad
\begin{bmatrix}
\sigma_{A_1} \\
\sigma_{A_2} \\
\sigma_{A_3} \\
\sigma_{A_4} \\
\sigma_{A_5} \\
\sigma_{A_6}
\end{bmatrix} = 
\begin{bmatrix}
0.0894 \\
0.0612 \\
0.0837 \\
0.0570 \\
0.0775 \\
0.0742
\end{bmatrix}
\quad
\begin{bmatrix}
\rho_{A_1} & \rho_{A_2} & \rho_{A_3} & \rho_{A_4} & \rho_{A_5} & \rho_{A_6}
\end{bmatrix} = 
\begin{bmatrix}
1.000 & 0.807 & 0.735 & 0.589 & 0.722 & 0.678 \\
0.807 & 1.000 & 0.621 & 0.716 & 0.634 & 0.623 \\
0.735 & 0.621 & 1.000 & 0.667 & 0.617 & 0.685 \\
0.589 & 0.716 & 0.667 & 1.000 & 0.961 & 0.920 \\
0.722 & 0.633 & 0.617 & 0.961 & 1.000 & 0.870 \\
0.678 & 0.623 & 0.685 & 0.920 & 0.870 & 1.000
\end{bmatrix}
$$
It should be noted that the problem has ten random variables defined as follows:

\[
\begin{align*}
A_1 &= \mu_{A_1} (1 + q_{A_1}) = \mu_1 (1 + q_1) \\
A_2 &= \mu_{A_2} (1 + q_{A_2}) = \mu_2 (1 + q_2) \\
A_3 &= \mu_{A_3} (1 + q_{A_3}) = \mu_3 (1 + q_3) \\
A_4 &= \mu_{A_4} (1 + q_{A_4}) = \mu_4 (1 + q_4) \\
A_5 &= \mu_{A_5} (1 + q_{A_5}) = \mu_5 (1 + q_5) \\
E &= \mu_E (1 + q_E) = \mu_7 (1 + q_7) \\
\alpha &= \mu_{\alpha} (1 + q_{\alpha}) = \mu_8 (1 + q_8) \\
P &= \mu_P (1 + q_P) = \mu_9 (1 + q_9) \\
T &= \mu_T (1 + q_T) = \mu_{10} (1 + q_{10})
\end{align*}
\]

Thus, the selected stochastic response for six-bar truss can be obtained as follows:

\[
\begin{bmatrix}
\mu_{F_1} \\
\mu_{F_4} \\
\mu_{x_2} \\
\mu_{x_3} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3}
\end{bmatrix}^T = 
\begin{bmatrix}
\mu_{F_1} \\
0.129kip \\
0.01091in. \\
-0.000182in. \\
2.182ksi \\
-0.0909ksi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mu_{F_1} \\
\mu_{F_4} \\
\mu_{x_2} \\
\mu_{x_3} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3}
\end{bmatrix}^H = 
\begin{bmatrix}
-1.090kip \\
0.127kip \\
0.01094in. \\
-0.000176in. \\
2.189ksi \\
-0.0880ksi
\end{bmatrix}
\]
\[
\begin{bmatrix}
\sigma_{F_1} \\
\sigma_{F_4} \\
\sigma_{x_2} \\
\sigma_{x_3} \\
\sigma_{\sigma_2} \\
\sigma_{\sigma_3}
\end{bmatrix} = \begin{bmatrix}
0.0529 kip \\
0.0746 kip \\
0.000553 in. \\
0.000970 in. \\
0.116 ksi \\
0.0968 ksi
\end{bmatrix} \quad \begin{bmatrix}
1.000 & -0.955 \\
-0.955 & 1.000 \\
1.000 & 0.520 \\
0.520 & 1.000 \\
1.000 & 0.466 \\
0.466 & 1.000
\end{bmatrix}.
\]

In the stochastic responses, it can be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of two bar forces are about 4.8% for the first bar and 57.8% for the fourth bar; the standard deviation of the displacements, \(x_2\) and \(x_3\), are 5.1% and 53.3%; the standard deviation of the stresses are about 5.3% for the second bar and 106.5% for the fifth bar.

The probability density function and cumulative distribution function for forces in the third and fourth bar are shown in Fig. 5.43(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description are almost equal to each other.

![Graph](image)

(a) Probability density functions for two forces.

Fig. 5.43 Stochastic description of forces in the third and fourth bar.
(b) Cumulative distribution functions for two forces.

Fig. 5.43 Stochastic description of forces in the third and fourth bar (Continued).

The selected response values for different probability levels of occurrence ($p = 50\%, 25\%$ and $75\%$) are listed in Table 5.11. In the table, about $3\%$ and $40.2\%$ changes are found in the first and fourth bar force in the $p = 25\%$ and $75\%$ levels; about $3.7\%$ and $36.9\%$ changes are found in displacements, $x_2$ and $x_3$, respectively; about $3.6\%$ and $37.5\%$ changes are found in the second and fifth bar stresses.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
<th>$p = 50%$</th>
<th>$p = 25%$</th>
<th>$p = 75%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column Force</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_1 (kip)$</td>
<td></td>
<td>-1.090</td>
<td>-1.054</td>
<td>-1.126</td>
</tr>
<tr>
<td>$F_4 (kip)$</td>
<td></td>
<td>0.127</td>
<td>0.0770</td>
<td>0.178</td>
</tr>
<tr>
<td>Displacements</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2 (in.)$</td>
<td></td>
<td>0.0109</td>
<td>0.0106</td>
<td>0.0113</td>
</tr>
<tr>
<td>$x_3 (in.)$</td>
<td></td>
<td>-0.000176</td>
<td>-0.000110</td>
<td>-0.000241</td>
</tr>
<tr>
<td>Column Stress</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_2 (ksi)$</td>
<td></td>
<td>2.189</td>
<td>2.110</td>
<td>2.267</td>
</tr>
<tr>
<td>$\sigma_5 (ksi)$</td>
<td></td>
<td>-0.088</td>
<td>-0.0552</td>
<td>-0.121</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.
The sensitivity analysis of the third bar force, displacement $x_1$ and stress in second bar, with respect to the random variables at the 75% probability of occurrence, is shown in Fig. 5.44(a), (b) and (c), respectively. Note that there is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are different. In the force sensitivity, the third bar force is sensitive to the load $P$, the Young’s modulus $E$, coefficient of thermal expansion $\alpha$, temperature $T$ and the second bar area $A_2$, not sensitive to bar areas $A_6$, $A_3$ and $A_5$. In the displacement sensitivity, the displacement, $x_1$, is the most sensitive to the first bar area $A_1$, not sensitive to bar areas $A_6$, $A_3$ and $A_5$. In the stress sensitivity, the second bar stress is sensitive to the second bar area $A_2$, the load $P$, the Young’s modulus $E$, coefficient of thermal expansion $\alpha$ and temperature $T$, not sensitive to bar areas $A_6$, $A_3$ and $A_5$.

(a) Sensitivity for the third bar force.

Fig. 5.44 Sensitivity analysis of response in the six-bar truss.
In stochastic design optimization, it is assumed that the material density has a mean of 0.289 \text{lbf/in.}^3 and standard deviation of 0.005 \text{lbf/in.}^3. The allowable strength has a mean of 2500 \text{psi} and standard deviation of 250 \text{psi}. The allowable horizontal and vertical displacement has a mean of 0.0025 \text{in.}, 0.012 \text{in.} and standard deviation of 0.00025 \text{in.}, 0.0012 \text{in.}, respectively. 0.001\text{in.}^2 \leq A_1, \ldots, A_6 \leq 30.0\text{in.}^2 The optimal results are shown in Fig. 5.45. The optimization results for both the first- and second-order
approximations in the stochastic analysis almost equal to each other, since the variances are small.

(a) Optimal weight versus the probability of occurrence $p$.

(b) Design variables versus the probability of occurrence $p$.

Fig. 5.45 Optimal results of the six-bar truss.
Example 12: A ring problem.

A uniform circular ring of radius $R$ is subjected to self-equilibrating forces as shown in Fig. 5.46(a). The sizing design variables, two moments of inertia $I_1$ and $I_2$, and two depths, $d_1$ and $d_2$, and two areas $A_1$ and $A_2$, have the following stochastic properties:

![Diagram of a ring](image1)

(a) Loads in a ring.

![Free-body diagram for the half ring](image2)

(b) Free-body diagram for the half ring.

![Transverse EE at A](image3)

(c) Transverse EE at $A$.

Fig. 5.46 A ring problem.
Three material variables, which consist of two elasticity modulii $E$, and $E_b$ and shear modulus $G$, have the stochastic properties:

$$
\begin{align*}
\{\mu_G\} & = \begin{bmatrix} 20,000 \\ 70,000 \end{bmatrix} ksi \\
\{\sigma_G\} & = \begin{bmatrix} 1,400 \\ 5,200 \end{bmatrix} ksi \\
\{\sigma_{\mu}\} & = \begin{bmatrix} 1,000 \\ 0.885 \\ 0.923 \end{bmatrix} \\
\{\mu_{\mu}\} & = \begin{bmatrix} 0.923 \\ 0.870 \\ 1.000 \end{bmatrix}
\end{align*}
$$

The load $P$ has a mean of $25.0$ kip and standard deviation of $2.5$ kip.

It should be noted that only half of the ring needs to be considered because of symmetry. Thus, the problem has ten random variables defined as,

$$
I_1 = \mu_{i1} (1 + q_{i1}) = \mu_i (1 + q_i) \\
I_2 = \mu_{i2} (1 + q_{i2}) = \mu_i (1 + q_i) \\
G = \mu_G (1 + q_G) = \mu_t (1 + q_t) \\
E_i = \mu_{E_i} (1 + q_{E_i}) = \mu_e (1 + q_e) \\
E_b = \mu_{E_b} (1 + q_{E_b}) = \mu_{E_b} (1 + q_{E_b}) \\
A_1 = \mu_{A_1} (1 + q_{A_1}) = \mu_s (1 + q_s) \\
A_2 = \mu_{A_2} (1 + q_{A_2}) = \mu_s (1 + q_s) \\
P = \mu_P (1 + q_P) = \mu_{10} (1 + q_{10})
$$

Therefore, the selected stochastic responses can be obtained as follows:

$$
\begin{align*}
\{\mu_{\sigma_{\mu}}\} & = \begin{bmatrix} 3.9789 \text{kip} \\ 312.500 \text{in.} - \text{kip} \\ 85.387 \text{in.} - \text{kip} \\ 0.00269 \text{in.} \\ -0.001235 \text{in.} \\ 4.340 \text{ksi} \\ 1.186 \text{ksi} \end{bmatrix} \\
\{\mu_{\sigma_{\mu}}\} & = \begin{bmatrix} 3.9789 \text{kip} \\ 312.500 \text{in.} - \text{kip} \\ 85.387 \text{in.} - \text{kip} \\ 0.00269 \text{in.} \\ -0.001235 \text{in.} \\ 4.340 \text{ksi} \\ 1.186 \text{ksi} \end{bmatrix}
\end{align*}
$$
In the stochastic responses, it can be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of the normal force at $A$ and two moments are about 10.0%; the standard deviation of the displacements, $v_1$ and $w_2$, are 14.6%; the standard deviation of the bending stresses are about 13.8% at $A$ and 14.4% at $B$.

The probability density function and cumulative distribution function for bending stresses are shown in Fig. 5.47(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description are almost equal to each other.

![Graph showing probability density functions for bending stresses at $A$ and $B$.](image-url)

(a) Probability density functions for bending stresses at $A$ and $B$.  
Fig.5.47 Stochastic description of the bending stresses at $A$ and $B$.  

\[
\begin{bmatrix}
\sigma_{N_A} \\
\sigma_{M_A} \\
\sigma_{M_B} \\
\sigma_{v_1} \\
\sigma_{w_2} \\
\sigma_{\sigma_1} \\
\sigma_{\sigma_2}
\end{bmatrix} = \begin{bmatrix}
0.3980kip \\
31.25in. - klb \\
8.552in. - klb \\
0.000393in. \\
0.0001809in. \\
0.600ksi \\
0.17ksi
\end{bmatrix}, \quad \begin{bmatrix}
\rho_{v_1}
\end{bmatrix} = \begin{bmatrix}
1.000 & 0.000 & 0.999 \\
1.000 & 0.000 & 0.998 \\
0.998 & 0.999 & 1.000 \\
1.000 & -1.000 & 0.000 \\
-1.000 & 1.000 & 0.000 \\
1.000 & 0.999 & 0.000 \\
0.999 & 1.000 & 0.000
\end{bmatrix}.
\]
(b) Cumulative distribution functions for bending stresses at $A$ and $B$.

Fig. 5.47 Stochastic description of the bending stresses at $A$ and $B$ (Continued).

The values of the selected responses for the probability levels of occurrence ($p = 50\%, 25\%$ and $75\%$) are listed in Table 5.12. In the table, about $6.7\%$ change is found in the normal force at $A$ in the $p = 25\%$ and $75\%$ levels, about $3.8\%$ and $6.8\%$ changes are found in two moments; about $9.9\%$ and $9.6\%$ changes are found in displacements, $v_1$ and $w_2$, respectively; about $9.3\%$ and $9.6\%$ changes are found in two bending stresses, respectively.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Column Force</td>
<td></td>
</tr>
<tr>
<td>$N_A$ (kip)</td>
<td>3.979</td>
</tr>
<tr>
<td>$M_A$ (kip)</td>
<td>312.515</td>
</tr>
<tr>
<td>$M_B$ (kip)</td>
<td>85.366</td>
</tr>
<tr>
<td>Displacements</td>
<td></td>
</tr>
<tr>
<td>$v_1$ (in.)</td>
<td>0.00272</td>
</tr>
<tr>
<td>$w_2$ (in.)</td>
<td>-0.00125</td>
</tr>
<tr>
<td>Column Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_1$ (ksi)</td>
<td>4.362</td>
</tr>
<tr>
<td>$\sigma_2$ (ksi)</td>
<td>1.193</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.
The sensitivity analysis of moment at $A$, vertical displacement at $A$ and bending stress at $B$, with respect to the random variables at the 75% probability of occurrence, is shown in Fig. 5.48(a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $A$ is sensitive to the load $P$ and moments of inertia $I_1$ and $I_2$, not sensitive to other random variables. The sensitivity of the displacement $x_A$ is sensitive to the loads $P$, Young’s modulus $E_b$ moments of inertia $I_1$ and $I_2$, not sensitive to other random variables.

![Graph of sensitivity analysis](image)

(a) Sensitivity for moment at $A$.

![Graph of sensitivity analysis](image)

(b) Sensitivity for displacement at $A$.

Fig.5.48 Sensitivity analysis of response in the ring.

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Fig. 5.48 Sensitivity analysis of response in the ring (Continued).

(c) Sensitivity for bending stress at B.

In the stress sensitivity, the bending stress at C is sensitive to the load $P$, depth $d_2$ and moments of inertia $I_1$ and $I_2$, not sensitive to other random variables.

In stochastic design optimization, the material density of the ring has a mean of 0.289 lbf/in.$^3$, and standard deviation of 0.005 lbf/in.$^3$. The permissible shear, tension and bending stresses have mean values of 200.0 psi, 100.0 psi and 4,000.0 psi, and standard deviations of 20.0 psi, 10.0 psi and 400.0 psi, respectively. The allowable horizontal and vertical displacements have mean values of 0.0015 in. and 0.001 in. and standard deviation of 0.0015 in. and 0.0001 in., respectively. $0.001in. \leq t \leq 12.0in$ and $0.001in. \leq d_1,d_2 \leq 24.0in$. As a result, the optimal expected weight and design variables are plotted in Fig. 5.49. The optimization results for both the first- and second-order approximations in the stochastic analysis almost equal to each other, since the variances are small.
Example 13: Three-span beam under a distributed load.

A three-span continuous beam, shown in Fig. 5.50, is made of a signal material of Young’s modulus $E$ with a mean of 10,000 ksi, and standard deviation of 141.421 ksi, and subjected to a uniformly distributed load $w$ with a mean of 7.5 kip/in. and standard deviation of 0.474 kip/in. Only half of the beam needs to be considered because of
symmetry. Thus, six random design variables, three moments of inertia and three depths, have the mean, standard deviation and correlation coefficient matrix as follows:

\[
\begin{bmatrix}
\mu_i \\
\mu_i \\
\mu_i \\
\mu_i \\
\mu_d \\
\mu_d \\
\mu_d \\
\end{bmatrix} = \begin{bmatrix}
864\text{in.}^4 \\
864\text{in.}^4 \\
864\text{in.}^4 \\
864\text{in.}^4 \\
12\text{in.} \\
12\text{in.} \\
12\text{in.} \\
\end{bmatrix} \quad \begin{bmatrix}
\sigma_i \\
\sigma_i \\
\sigma_i \\
\sigma_i \\
\sigma_d \\
\sigma_d \\
\sigma_d \\
\end{bmatrix} = \begin{bmatrix}
55.994\text{in.}^3 \\
57.959\text{in.}^3 \\
64.656\text{in.}^3 \\
0.679\text{in.} \\
0.720\text{in.} \\
0.759\text{in.} \\
\end{bmatrix} \quad \begin{bmatrix}
1.000 & 0.805 & 0.742 \\
0.805 & 1.000 & 0.697 \\
0.742 & 0.697 & 1.000 \\
\end{bmatrix}
\]

(a) Three-span beam.

(b) Free-body diagram for the half beam.

Fig. 5.50 Three-span beam under distributed load.

Therefore, the problem for half of beam has eight random variables defined as:

\[
\begin{align*}
I_1 &= \mu_i (1 + q_i) = \mu_i (1 + q_1) \\
I_2 &= \mu_i (1 + q_i) = \mu_i (1 + q_2) \\
I_3 &= \mu_i (1 + q_i) = \mu_i (1 + q_3) \\
d_1 &= \mu_d (1 + q_d) = \mu_d (1 + q_4) \\
d_2 &= \mu_d (1 + q_d) = \mu_d (1 + q_5) \\
d_3 &= \mu_d (1 + q_d) = \mu_d (1 + q_6) \\
d_4 &= \mu_d (1 + q_d) = \mu_d (1 + q_7) \\
d_5 &= \mu_d (1 + q_d) = \mu_d (1 + q_8) \\
E &= \mu_E (1 + q_E) = \mu_E (1 + q_f) \\
w &= \mu_w (1 + q_w) = \mu_w (1 + q_k)
\end{align*}
\]
Hence, the selected responses can be obtained as follows:

\[
\begin{bmatrix}
\mu_{\pi_2} \\
\mu_{\pi_4} \\
\mu_{\pi_E} \\
\mu_{\sigma_8} \\
\mu_{\sigma} \\
\mu_{\sigma_6}
\end{bmatrix} = \begin{bmatrix}
\mu_{M_2} \\
\mu_{M_4} \\
\mu_\nu \\
\mu_{\theta_8} \\
\mu_{\sigma_3} \\
\mu_{\sigma_6}
\end{bmatrix}^T = \begin{bmatrix}
3750.000 \text{ in.} - \text{klb} \\
\mu_{M_2} \\
\mu_{M_4} \\
\mu_\nu \\
\mu_{\theta_8} \\
\mu_{\sigma_3}
\end{bmatrix} ^T = \begin{bmatrix}
3750.022 \text{ in.} - \text{klb} \\
\mu_{M_2} \\
\mu_{M_4} \\
\mu_\nu \\
\mu_{\theta_8} \\
\mu_{\sigma_3}
\end{bmatrix} \\
= \begin{bmatrix}
\mu_{\pi_2} \\
\mu_{\pi_4} \\
\mu_{\pi_E} \\
\mu_{\sigma_8} \\
\mu_{\sigma} \\
\mu_{\sigma_6}
\end{bmatrix} = \begin{bmatrix}
237.221 \text{ in.} - \text{klb} \\
118.982 \text{ in.} - \text{klb} \\
0.00330 \text{ in.} \\
0.827 \times 10^{-4} \text{ rad} \\
2.865 \text{ ksi} \\
2.301 \text{ ksi}
\end{bmatrix} = \begin{bmatrix}
1.000 \\
0.995 \\
0.995 \\
1.000 \\
0.978 \\
1.000
\end{bmatrix} \begin{bmatrix}
0.995 \\
1.000 \\
0.978 \\
1.000 \\
0.853 \\
1.000
\end{bmatrix} = \begin{bmatrix}
0.995 \\
1.000 \\
0.853 \\
1.000
\end{bmatrix}
\]

In the stochastic responses, it can be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of two moments are about 6.3%; the standard deviation of the displacement \(v_E\) is 9.0%, the standard deviation of rotation \(\theta_E\) is 9.1%; the standard deviation of the bending stresses are about 11.0% at \(E\) and 11.8% at \(F\), respectively.

The probability density function and cumulative distribution function for bending stresses at \(E\) and \(F\) are shown in Fig. 5.51. It is obvious that both the first- and second-order approximations in the probabilistic description have very small differences.
The calculated responses for different probability levels of occurrence ($p = 50\%$, $25\%$ and $75\%$) are listed in Table 5.13. In the table, it may be noted that about $4.3\%$ percent change is found in the two moments in the $p = 25\%$ and $75\%$ levels; about $6.0\%$
change is found in the displacement at $E$, about 6.2\% change is found in the rotation at $B$, whereas about 7.2\%, and 7.9\% changes are noted in the two bending tresses.

Table 5.13 Response values for p-percent probability of success in the three-span beam.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Moment</td>
<td></td>
</tr>
<tr>
<td>$M_2$ (in.-klb)</td>
<td>3750.022</td>
</tr>
<tr>
<td>$M_4$ (in.-klb)</td>
<td>-1874.956</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
</tr>
<tr>
<td>$v_E$ (in.)</td>
<td>-0.0369</td>
</tr>
<tr>
<td>$\theta_B$ (rad)</td>
<td>-0.000908</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_3$ (ksi)</td>
<td>26.151</td>
</tr>
<tr>
<td>$\sigma_6$ (ksi)</td>
<td>19.644</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of moment at $E$, displacement at $F$ and bending stress at $B$, with respect to the random variables at the 75\% probability of occurrence, is shown in Fig. 5.52(a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at $E$ is the most sensitive to the distributed load $w$, moments of inertia $I_1$ and $I_2$, not sensitive to other random variables. The sensitivity of the displacement $x_F$ is the most sensitive to the distributed load $w$, the Young’s modulus $E$ and moments of inertia $I_1$, $I_2$ and $I_3$, not sensitive to the three depths. In the stress sensitivity, the bending stress at $B$ is sensitive to moments of inertia $I_2$, the depth $d_2$ and the distributed load $w$, not sensitive to the two depths, $d_1$ and $d_3$, and the Young’s modulus $E$. 
Fig. 5.52 Sensitivity analysis of response in the three-span beam.

(a) Sensitivity for moment at E.

(b) Sensitivity for the displacement at F.

(c) Sensitivity for the bending stress at B.

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In stochastic design optimization, it is assumed that the material density has a mean of 0.289 \( \text{lbf/in.}^3 \), and standard deviation of 0.005 \( \text{lbf/in.}^3 \). The allowable strength has a mean of 16.0 \( \text{ksi} \) and standard deviation of 1.6 \( \text{ksi} \). The allowable displacement has a mean of 0.025 \( \text{in.} \) and standard deviation of 0.0025 \( \text{in.} \). \( 0.001 \text{in.} \leq d_1, d_2, d_3 \leq 24.0 \text{in.} \) As a result, the optimal results are shown in Fig. 5.53. The optimization results for both the first- and second-order approximations in the stochastic analysis almost equal to each other, since the variances are small.

(a) Optimal weight versus the probability of occurrence \( p \).

(b) Design variables versus the probability of occurrence \( p \).

Fig. 5.53 Optimal results of a three-span beam.
Example 14: Portal frame.

A steel portal frame has the geometrical dimensions and load shown in Fig. 5. 54(a). The Young’s modulus $E$ has a mean of 30,000 ksi and standard deviation of 2,078.461 ksi. The concentrated load $P$ has a mean of 18.0 kip and standard deviation of 1.8 kip. The six random design variables, three moments of inertia and three depths, have the following stochastic properties:

$$
\begin{align*}
\{ \mu_{t_1} \} &= \{ 260in.^4 \} \\
\{ \mu_{I_2} \} &= \{ 360in.^4 \} \\
\{ \mu_{I_3} \} &= \{ 390in.^4 \} \\
\{ \mu_{d_1} \} &= \{ 10in. \} \\
\{ \mu_{d_2} \} &= \{ 10in. \} \\
\{ \mu_{d_3} \} &= \{ 10in. \}
\end{align*}
$$

$$
\begin{align*}
\{ \sigma_{t_1} \} &= \{ 14.241in.^3 \} \\
\{ \sigma_{I_2} \} &= \{ 19.049in.^3 \} \\
\{ \sigma_{I_3} \} &= \{ 19.500in.^3 \} \\
\{ \sigma_{d_1} \} &= \{ 0.6in. \} \\
\{ \sigma_{d_2} \} &= \{ 0.6in. \} \\
\{ \sigma_{d_3} \} &= \{ 0.6in. \}
\end{align*}
$$

$$\begin{bmatrix}
0.000 & 0.552 & 0.730 \\
0.552 & 1.000 & 0.907 \\
0.730 & 0.907 & 1.000
\end{bmatrix}
$$

It is obvious that the problem has eight random variables defined as follows:

$$
\begin{align*}
I_1 &= \mu_{t_1}(1 + q_{t_1}) = \mu_{I_1}(1 + q_{I_1}) \\
I_2 &= \mu_{I_2}(1 + q_{I_2}) = \mu_{I_2}(1 + q_{I_2}) \\
I_3 &= \mu_{I_3}(1 + q_{I_3}) = \mu_{I_3}(1 + q_{I_3}) \\
E &= \mu_{E}(1 + q_{E}) = \mu_{E}(1 + q_{E}) \\
P &= \mu_{P}(1 + q_{P}) = \mu_{P}(1 + q_{P})
\end{align*}
$$

Thus, the selected responses can be given as follows:

$$
\begin{align*}
\{ \mu_{\sigma_{t_1}} \} &= \{ 136.137in. - kib \} \\
\{ \mu_{\sigma_{I_2}} \} &= \{ 617.939in. - kib \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ 0.1227in. \} \\
\{ \mu_{\theta_{d_1}} \} &= \{ -0.00154rad \} \\
\{ \mu_{\theta_{d_2}} \} &= \{ 8.582ksi \}
\end{align*}
$$

$$
\begin{align*}
\{ \mu_{\sigma_{t_2}} \} &= \{ 1.115ksi \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ 136.113in. - kib \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ 617.951in. - kib \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ 0.1236in. \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ -0.00155rad \} \\
\{ \mu_{\sigma_{I_3}} \} &= \{ 8.606ksi \}
\end{align*}
$$

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Fig. 5.54 Analysis of the portal frame.
\[
\begin{bmatrix}
\sigma_{M_d} \\
\sigma_{M_c} \\
\sigma_{X_s} \\
\sigma_{\theta_b} \\
\sigma_{\sigma_c} \\
\sigma_{\sigma_E}
\end{bmatrix} = \begin{bmatrix}
13.789 \text{in. - klb} \\
61.902 \text{in. - klb} \\
0.01644 \text{in.} \\
0.000201 \text{rad} \\
1.091 \text{ksi} \\
0.150 \text{ksi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.000 & 0.976 \\
0.976 & 1.000 \\
1.000 & -0.974 \\
-0.974 & 1.000 \\
1.000 & 0.780 \\
0.780 & 1.000
\end{bmatrix}
\]

In the stochastic responses, it can be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviation of two moments are about 10.1% and 10.0%, respectively; the standard deviation of the displacement \(X_s\), is 13.4%, the standard deviation of rotation \(\theta_b\) is 13.1%; the standard deviation of the bending stresses are about 12.7% at \(C\) and 13.5% at \(E\), respectively.

The probability density function and cumulative distribution function for displacements, \(X_s\) and \(X_c\), are shown in Fig. 5.55(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description have very small differences.

![Probability density functions for displacements, \(X_s\) and \(X_c\).](image)

(a) Probability density functions for displacements, \(X_s\) and \(X_c\).

Fig. 5.55 Stochastic description of the displacements, \(X_s\) and \(X_c\).
(b) Cumulative distribution functions for displacements, $X_s$ and $X_c$.

Fig. 5.55 Stochastic description of the displacements, $X_s$ and $X_c$ (Continued).

The evaluated responses for different probability levels of occurrence ($p = 50\%$, $25\%$ and $75\%$) are listed in Table 5.14. In the table, it may be noted that about 6.8\% percent changes are found in the two moments in the $p = 25\%$ and $75\%$ levels; about 9.0\% change is found in the displacements $X_s$, about 9.0\% change is found in the rotation at $B$, whereas about 8.5\%, and 9.0\% changes are noted in the two bending tresses.

Table 5.14 Response values for $p$-percent probability of success in the frame.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Moment</td>
<td></td>
</tr>
<tr>
<td>$M_A$ (in.-klb)</td>
<td>136.113</td>
</tr>
<tr>
<td>$M_C$ (in.-klb)</td>
<td>617.951</td>
</tr>
<tr>
<td>Displacement/Rotation</td>
<td></td>
</tr>
<tr>
<td>$X_s$ (in.)</td>
<td>0.1236</td>
</tr>
<tr>
<td>$\theta_B$ (rad)</td>
<td>-0.00155</td>
</tr>
<tr>
<td>Bending Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c$ (ksi)</td>
<td>8.606</td>
</tr>
<tr>
<td>$\sigma_E$ (ksi)</td>
<td>1.118</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.
The sensitivity analysis of moment at \( C \), displacement \( X_c \), and bending stress at \( A \), with respect to the random variables at the 75% probability of occurrence, is shown in Fig. 5.56(a), (b) and (c), respectively. There is a small difference between the deterministic sensitivity and stochastic sensitivity of the first-/second-order approximation. Both stochastic sensitivities are almost equal to each other, but sensitivities with respect to some variables are little different. In the moment sensitivity, the moment at \( C \) is sensitive to the load \( P \), three moments of inertia \( I_1, I_2 \) and \( I_3 \), not sensitive to other random variables. The sensitivity of the displacement \( X_c \) is sensitive to the load \( P \), the Young’s modulus \( E \) and moment of inertia \( I_1 \), not sensitive to the three depths. In the stress sensitivity, the bending stress at \( A \) is sensitive to the depth \( d_1 \), the load \( P \) and moment of inertia \( I_1 \), not sensitive to the two depths, \( d_2 \) and \( d_3 \), and the Young’s modulus \( E \).

![Diagram](image)

(a) Sensitivity for moment at \( C \).

Fig. 5.56 Sensitivity analysis of response in the frame.
In stochastic design optimization, it is assumed that the material density has a mean of 0.289 \textit{lbf/in.}^3, and standard deviation of 0.002 \textit{lbf/in.}^3. The allowable strength has a mean of 3.20 \textit{ksi} and standard deviation of 0.32 \textit{ksi}. The allowable displacement has a mean of 0.08 \textit{in.} and standard deviation of 0.008 \textit{in.} \text{0.1in.} \leq d_1, d_2, d_3 \leq 40.0\text{in.} As a result, the optimal results are shown in Fig. 5.57. The optimization results for both the
first- and second-order approximations in the stochastic analysis almost equal to each other, since the variances are small.

(a) Optimal weight versus the probability of occurrence $p$.

(b) Design variables versus the probability of occurrence $p$.

Example 15: Navier’s table problem.

A symmetrical table with four legs is subjected to a concentrated load $P$ ($\mu_p = 10.0\text{kip.}$ and $\sigma_p = 0.632\text{kip}$) acting at eccentric place, $e_x = 0.2\text{in.}$ and $e_y = 0.1\text{in.}$, as shown in Fig. 5.58. The distances between the legs along the $x$- and $y$- directions are $2a$ and $2b$ ($a = 20.0\text{in.}$, and $b = 25.0\text{in.}$), respectively. The sizing design variables, four
Fig. 5.58 Indeterminate table problem.

areas $A_1$, $A_2$, $A_3$ and $A_4$, have the mean, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{A_1} \\
\mu_{A_2} \\
\mu_{A_3} \\
\mu_{A_4}
\end{bmatrix} = 
\begin{bmatrix}
1.0 \\
1.0 \\
1.0 \\
1.0
\end{bmatrix}
\begin{bmatrix}
\sigma_{A_1} \\
\sigma_{A_2} \\
\sigma_{A_3} \\
\sigma_{A_4}
\end{bmatrix} = 
\begin{bmatrix}
0.0592 \\
0.0600 \\
0.0616 \\
0.0632
\end{bmatrix}
\begin{bmatrix}
\rho_{A_1A_2} \\
\rho_{A_1A_3} \\
\rho_{A_1A_4} \\
\rho_{A_2A_3} \\
\rho_{A_2A_4} \\
\rho_{A_3A_4} \\
\rho_{A_4A_2} \\
\rho_{A_4A_3}
\end{bmatrix} = 
\begin{bmatrix}
1.000 & 0.902 & 0.960 & 0.962 \\
0.902 & 1.000 & 0.811 & 0.922 \\
0.960 & 0.811 & 1.000 & 0.923 \\
0.962 & 0.922 & 0.923 & 1.000
\end{bmatrix}.
$$

The stochastic properties of the four random material variables are as following:

$$
\begin{bmatrix}
\mu_{E_1} \\
\mu_{E_2} \\
\mu_{E_3} \\
\mu_{E_4}
\end{bmatrix} = 
\begin{bmatrix}
30,000 \\
30,000 \\
30,000 \\
30,000
\end{bmatrix}
\begin{bmatrix}
\sigma_{E_1} \\
\sigma_{E_2} \\
\sigma_{E_3} \\
\sigma_{E_4}
\end{bmatrix} = 
\begin{bmatrix}
2,012.461 \\
2,224.860 \\
2,284.732 \\
2,323.790
\end{bmatrix}
\begin{bmatrix}
\rho_{E_1E_2} \\
\rho_{E_1E_3} \\
\rho_{E_1E_4} \\
\rho_{E_2E_3} \\
\rho_{E_2E_4} \\
\rho_{E_3E_4}
\end{bmatrix} = 
\begin{bmatrix}
1.000 & 0.925 & 0.940 & 0.962 \\
0.925 & 1.000 & 0.815 & 0.836 \\
0.940 & 0.815 & 1.000 & 0.780 \\
0.962 & 0.836 & 0.780 & 1.000
\end{bmatrix}.
$$

We note that the problem has nine random variables described as:

$$
\begin{align*}
A_1 &= \mu_{A_1}(1+q_{A_1}) = \mu_1(1+q_1) & E_2 &= \mu_{E_2}(1+q_{E_2}) = \mu_6(1+q_6) \\
A_2 &= \mu_{A_2}(1+q_{A_2}) = \mu_2(1+q_2) & E_3 &= \mu_{E_3}(1+q_{E_3}) = \mu_7(1+q_7) \\
A_3 &= \mu_{A_3}(1+q_{A_3}) = \mu_3(1+q_3) & E_4 &= \mu_{E_4}(1+q_{E_4}) = \mu_8(1+q_8) \\
A_4 &= \mu_{A_4}(1+q_{A_4}) = \mu_4(1+q_4) & P &= \mu_{P}(1+q_{P}) = \mu_9(1+q_9) \\
E_1 &= \mu_{E_1}(1+q_{E_1}) = \mu_5(1+q_5)
\end{align*}
$$
Hence, the stochastic responses can be obtained as follows:

**Force:**
\[
\begin{align*}
\begin{bmatrix}
\mu_{F_1} \\
\mu_{F_2} \\
\mu_{F_3} \\
\mu_{F_4}
\end{bmatrix}
&= \begin{bmatrix}
\mu_{F_1}' \\
\mu_{F_2}' \\
\mu_{F_3}' \\
\mu_{F_4}'
\end{bmatrix}
\begin{bmatrix}
-2.4588 \\
-2.4787 \\
-2.5413 \\
-2.5212
\end{bmatrix}_{kip} \\
\begin{bmatrix}
\sigma_{F_1} \\
\sigma_{F_2} \\
\sigma_{F_3} \\
\sigma_{F_4}
\end{bmatrix}
&= \begin{bmatrix}
0.162 \\
0.164 \\
0.167 \\
0.166
\end{bmatrix}_{kip}
\begin{bmatrix}
1.000 & 0.836 & 1.000 & 0.839 \\
0.836 & 1.000 & 0.841 & 1.000 \\
0.839 & 1.000 & 0.844 & 1.000
\end{bmatrix}.
\end{align*}
\]

**Displacement:**
\[
\begin{align*}
\begin{bmatrix}
\mu_\pi \\
\mu_\theta_x \\
\mu_\theta_y
\end{bmatrix}
&= \begin{bmatrix}
\mu_w' \\
\mu_{\theta_x}' \\
\mu_{\theta_y}'
\end{bmatrix}
\begin{bmatrix}
0.0125 \times 10^{-5} \text{ rad} \\
-0.8000 \times 10^{-6} \text{ rad} \\
-0.3137 \times 10^{-5}
\end{bmatrix}
\begin{bmatrix}
1.000 & 0.01009 \text{ in.} \\
0.019 & 1.000 & 0.407 \\
0.019 & 0.407 & 1.000
\end{bmatrix}.
\end{align*}
\]

**Stress:**
\[
\begin{align*}
\begin{bmatrix}
\mu_{\sigma_1} \\
\mu_{\sigma_2} \\
\mu_{\sigma_3} \\
\mu_{\sigma_4}
\end{bmatrix}
&= \begin{bmatrix}
\mu_{\sigma_1}' \\
\mu_{\sigma_2}' \\
\mu_{\sigma_3}' \\
\mu_{\sigma_4}'
\end{bmatrix}
\begin{bmatrix}
-2.4588 \\
-2.4787 \\
-2.5413 \\
-2.5212
\end{bmatrix}_{ksi} \\
\begin{bmatrix}
\sigma_{\sigma_1} \\
\sigma_{\sigma_2} \\
\sigma_{\sigma_3} \\
\sigma_{\sigma_4}
\end{bmatrix}
&= \begin{bmatrix}
0.217 \\
0.215 \\
0.224 \\
0.228
\end{bmatrix}_{ksi}
\begin{bmatrix}
1.000 & 0.934 & 0.980 & 0.925 \\
0.934 & 1.000 & 0.923 & 0.963 \\
0.980 & 0.923 & 1.000 & 0.940 \\
0.925 & 0.963 & 0.940 & 1.000
\end{bmatrix}.
\end{align*}
\]

In the stochastic responses, it may be noted that there is a very little difference between the second-order approximation mean values and deterministic solutions or the first-order approximation mean values. However, the standard deviations of four forces are about 6.6%; the standard deviation of the displacement \(w\) is 11.1%; the standard deviations of the bending stresses are about 9.0%.
The probability density function and cumulative distribution function for displacement, \( w \), are shown in Fig. 5.59(a) and (b), respectively. It is obvious that both the first- and second-order approximations in the probabilistic description have a small difference.

![Probability Density Function](image1)

(a) Probability density function for the displacement, \( w \).

![Cumulative Distribution Function](image2)

(b) Cumulative distribution function for the displacement, \( w \).

Fig. 5.59 Stochastic description of the stochastic displacement, \( w \).

The calculated response values for different probability levels of occurrence (\( p = 50\%, 25\% \) and \( 75\% \)) are listed in Table 5.15. In the table, about 4.4\% to 4.9\% changes
are found in the first to fourth bar force in the $p = 25\%$ and $75\%$ levels; about $7.4\%$
change is found in the displacement $w$, about $73.1\%$ and $61.9\%$ changes are found in the
rotations; about $5.8\%$ to $6.1\%$ changes are found in the first to fourth bar stresses.

Table 5.15 Response values for p-percent probability of success in the Navier’s table.

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Upper/Lower limit of response variable (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 50%$</td>
</tr>
<tr>
<td>Column Force</td>
<td></td>
</tr>
<tr>
<td>$F_1 \ (kip)$</td>
<td>-2.4594</td>
</tr>
<tr>
<td>$F_2 \ (kip)$</td>
<td>-2.4781</td>
</tr>
<tr>
<td>$F_3 \ (kip)$</td>
<td>-2.5419</td>
</tr>
<tr>
<td>$F_4 \ (kip)$</td>
<td>-2.5206</td>
</tr>
<tr>
<td>Displacements</td>
<td></td>
</tr>
<tr>
<td>$w \ (in.)$</td>
<td>0.01009</td>
</tr>
<tr>
<td>$\theta_x \ (rad)$</td>
<td>0.3282×10^{-5}</td>
</tr>
<tr>
<td>$\theta_y \ (rad)$</td>
<td>-0.8077×10^{-6}</td>
</tr>
<tr>
<td>Column Stress</td>
<td></td>
</tr>
<tr>
<td>$\sigma_1 \ (ksi)$</td>
<td>-2.4679</td>
</tr>
<tr>
<td>$\sigma_2 \ (ksi)$</td>
<td>-2.4864</td>
</tr>
<tr>
<td>$\sigma_3 \ (ksi)$</td>
<td>-2.5512</td>
</tr>
<tr>
<td>$\sigma_4 \ (ksi)$</td>
<td>-2.5305</td>
</tr>
</tbody>
</table>

Note: Lower value if mean is negative; Expressed as percent of mean.

The sensitivity analysis of force at the first leg, displacement $w$ and stress at the
fourth leg, with respect to the random variable at the $75\%$ probability of occurrence, is
shown in Fig. 5.60(a), (b) and (c), respectively. Note that there is a small difference
between the deterministic sensitivity and stochastic sensitivity of the first-/second-order
approximation. Both stochastic sensitivities are almost equal to each other, but
sensitivities with respect to some variables are different. In the force sensitivity, the first
bar force is the most sensitive to the load $P$, the bar areas and the Young’s modulus have
the equal effects on the first bar force. In the displacement sensitivity, the displacement $w$
is similar to the sensitivity of the first bar force, but all bar areas and the Young’s
modulus have the equal minus effects on the displacement. In the stress sensitivity, the
(a) Sensitivity for force at the first leg.

(b) Sensitivity for displacement $w$.

(c) Sensitivity for bending stress at the fourth leg.

Fig. 5.60 Sensitivity analysis of responses in the Navier’s table.
fourth bar stress is the most sensitive to the fourth bar area $A_4$ and the load $P$, other random variables have similar to the sensitivity of the first bar force.

In stochastic design optimization, it is assumed that the material density has a mean of 0.289 $\text{lbf/in.}^3$ and standard deviation of 0.005 $\text{lbf/in.}^3$. The allowable strength has a mean of 2.0 $\text{ksi}$ and standard deviation of 0.20 $\text{ksi}$. The displacement limitation has a mean of 0.008 $\text{in.}$ and standard deviation of 0.0008 $\text{in.}$. As a result, the optimal results are shown in Fig. 5.61. The optimization results for both the first- and second-order approximations in the stochastic analysis almost equal to each other, since the variances are small.

![Graph showing optimal weight versus probability](image1)

(a) Optimal weight versus the probability of occurrence $p$.

![Graph showing design variables versus probability](image2)

(b) Design variables versus the probability of occurrence $p$.

Fig. 5.61 Optimal results of the Navier’s table.
5.2.2 Verification by Monte Carlo simulations in Example 10

Table 5.16 Comparison of stochastic analysis with Monte Carlo Simulations.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Integrate Force Method</th>
<th>Stiffness Method (ANSYS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Perturbation</td>
<td>DMCS (20000)</td>
</tr>
<tr>
<td></td>
<td>µ^I</td>
<td>µ</td>
</tr>
<tr>
<td></td>
<td>µ^II</td>
<td>µ</td>
</tr>
<tr>
<td></td>
<td>X (in.)</td>
<td>0.197</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.237</td>
</tr>
<tr>
<td></td>
<td>61.215</td>
<td>61.702</td>
</tr>
<tr>
<td>Time (s)</td>
<td>4.999</td>
<td>7.029</td>
</tr>
<tr>
<td>Normalizing Time(s)</td>
<td>1</td>
<td>1.406</td>
</tr>
</tbody>
</table>

In the case of three-bar truss, the mean values and standard deviations of force, displacement and stress are computed using the first- and second-order approximation of stochastic analysis for IFM. In Monte Carlo simulations, direct Monte Carlo simulation and Latin Hypercube sampling of IFM are used to calculate the statistic values of responses with 20,000 and 5,000 realizations, respectively. In the stiffness method, the mean and standard deviation of responses are simulated by using direct Monte Carlo simulation and Latin Hypercube sampling in ANSYS with 12,500 and 1,000 realizations, respectively. These computations and simulations are performed on a Pentium® 4 CPU 3.2GHz computer.

It should be seen that the agreement of mean values are better than that of standard deviations; the stochastic analysis solutions with the second-order approximation compare well with direct MCS solutions; the displacement’s mean values and standard deviations are in fairly good agreement with that of MCS of both IFM and SM. Although all methods agree well and are evidently comparable in accuracy, the
stochastic analysis is the most efficient solution procedure. The following pictures are shown parts of results from the stiffness method by using ANSYS.

Fig. 5.62 Probability density and cumulative distribution function from MCS for the first bar force.

Fig. 5.63 Probability density and cumulative distribution function from MCS for the horizontal displacement.
Fig. 5.64 Probability density and cumulative distribution function from LHS for the third bar force.

Fig. 5.65 Probability density and cumulative distribution function from LHS for the vertical displacement.

5.2.3 Comparison with Neumann Expansion technique in Example 10

From Fig. 5.66 and 5.67, it is well understood that 1,500 realizations are sufficient for studying the trend in convergence and accuracy of NE-MCS results for this problem. The expectation and standard deviation of forces from NE-MCS, direct MCS and the
second-order perturbation method are presented in the Table 5.17. The mean values and standard deviations of forces obtained from NE-MCS technique are compared at $\mu_E = 30,000 ksi$ and $\sigma_E = 3,000 ksi$ with results obtained by direct MCS with 5,000 realizations and the second-order perturbation method. It should be noted that the expected value and standard deviation of forces decrease/increase with the order of expansion increasing and converge after the fifth order of expansion. It is also revealed that rate of convergence is substantially improved when the order of expansion in the NE-MCS series is increased from 1 to 2. After the third-order expansion, results tend to those obtained from direct simulation with gradually slower convergence rate. The rate of convergence of the standard deviation is slower than that of the mean value. The is due to the fact that the standard deviation is influenced by only deviatoric component of sample response, and the force has small sensitivity to the Young’s modulus. The CPU time increases as the order of expansion increases and the second-order perturbation method has least time.

![Fig. 5.66 Expectation fluctuation of the first bar force.](image)

\[\text{c. o. v. of } E = 0.05\]

\[\begin{align*}
\text{Expectation of } F^1
\end{align*}\]

\[\begin{align*}
\text{Number of Simulation}
\end{align*}\]
Fig. 5.67 Fluctuation of standard deviation of the first bar force.

Table 5.17 Statistical convergence study of forces.

<table>
<thead>
<tr>
<th>Order of NE-MCS (1500)</th>
<th>MCS (5000)</th>
<th>2nd-order PM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6 7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>61.21060 61.21291 61.21351 61.21353 61.21353 61.21353 61.21421 61.21518</td>
<td>61.21421 61.21518</td>
<td>( \mu )</td>
</tr>
<tr>
<td>( \sigma )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.041569 0.040778 0.041075 0.041116 0.041119 0.041119 0.041297 0.041177</td>
<td>0.041297 0.041177</td>
<td>( \sigma )</td>
</tr>
<tr>
<td>0.058788 0.057669 0.058089 0.058147 0.058151 0.058151 0.058402 0.058234</td>
<td>0.058402 0.058234</td>
<td>( \sigma )</td>
</tr>
<tr>
<td>0.041569 0.040778 0.041075 0.041116 0.041119 0.041119 0.041297 0.041177</td>
<td>0.041297 0.041177</td>
<td>( \sigma )</td>
</tr>
<tr>
<td>Time (s)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>41.390 57.515 93.420 106.044 136.140 175.363 225.221 81.156</td>
<td>81.156 5.889</td>
<td>Time (s)</td>
</tr>
</tbody>
</table>

The computation results of the expected values and standard deviations of the first bar force and the horizontal displacement \( x_1 \) by five methods (the first- and second-order perturbation method, NE-MCS with 100 realizations, direct MCS with 500 realizations and MCS with 500 realizations by using the stiffness method from ANSYS are shown in Fig. 5.68 to 5.71. It is noted that in the range of small variance of the Young’s modulus \( E \), the results of these methods are close. In the Fig. 5.68 and 5.70, a moderate increase is observed for the expected value of the first bar force and the horizontal displacement \( x_1 \).
Each expected force curve in Fig. 5.68 is shown to increase linearly, except for the first- and second-order perturbation methods and MCS from ANSYS, in which case the first- and second-order perturbation curves overlap each other and are constant, the MCS curve from ANSYS remains also constant and more close to the perturbation curves; however, each displacement curve in Fig. 5.70 has an accelerated rate of increase except for the first-order perturbation curve, in which case the expected value remains constant, the second perturbation curve is more close to the MCS curve from the stiffness method (ANSYS). In Fig. 5.69 and 5.71, the standard deviation curves of both the first bar force and horizontal displacement $x_1$ linearly increase with the coefficient of variation $E$ and are very close to each other. The NE-MCS curve is more close to MCS curve from IFM and the perturbation curves is more close to MCS curve from stiffness method.

It is important to note that the perturbation methods underestimate the response variability for large values of the coefficient of variation $E$, whereas the NE-MCS overestimate the response variability. It is consistent with the earlier results as demonstrated in some paper [58].

![Fig. 5.68 Comparison of expectation of the first bar force.](image)

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Fig. 5.69 Comparison of standard deviation of the first bar force.

Fig. 5.70 Comparison of expectation of the horizontal displacement, $x_1$. 
5.2.4 Comparison of different distributions with perturbation method

In the previous chapters, the random variables in structural parameters are defined as the homogenous Gauss process. To assess the efficiency and effectiveness of the perturbation method, stochastic responses in Example 10 are also simulated using normal, uniform, logistic, gamma and exponential distributions, respectively, and solved by the first- and second-order perturbation method in the Table 5.18. The mean value and standard deviation of Young’s modulus are assumed as 30,000 ksi and 3,000 ksi, respectively. In direct MCS, only one random variable, Young’s modulus $E$, is considered and simulated with 20,000 realizations using above five distributions, respectively. It should be noted that the first three distributions results are in good agreement with the second-order perturbation results on both expected values and standard deviations of responses. The force response obtained from SM in the gamma distribution has a small error with that of IFM due to the different moment parameters in
SM. However, the displacement has a good agreement with IFM. And the exponential distribution shows the bad agreement on both statistic values. So, if the random variables are not Gaussian process, one must evaluate their expressions with higher order moments accordingly.

Table 5.18 Comparison of different distributions with perturbation method.

<table>
<thead>
<tr>
<th>Methods/ Distributions</th>
<th>Responses</th>
<th>Forces $\mu$</th>
<th>Forces $\sigma$</th>
<th>Displacements $\mu$</th>
<th>Displacements $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Perturbation Method (IFM)</strong></td>
<td>1st-order Approximation</td>
<td>62.7804</td>
<td>0.08241</td>
<td>0.1975</td>
<td>0.02239</td>
</tr>
<tr>
<td></td>
<td></td>
<td>61.2152</td>
<td>0.08241</td>
<td>-0.2371</td>
<td>0.02002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-7.9303</td>
<td>0.1165</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2nd-order Approximation</td>
<td>62.7804</td>
<td>0.08241</td>
<td>0.1977</td>
<td>0.02002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>61.2152</td>
<td>0.08241</td>
<td>-0.2371</td>
<td>0.02002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-7.9303</td>
<td>0.1165</td>
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<tr>
<th></th>
<th><strong>Normal Distribution</strong></th>
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<tr>
<td>IM</td>
<td>62.7809</td>
<td>0.08239</td>
<td>0.1999</td>
<td>0.02334</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61.2145</td>
<td>0.1165</td>
<td>-0.2392</td>
<td>0.02087</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-7.9298</td>
<td>0.08239</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>62.7790</td>
<td>0.08357</td>
<td>0.1995</td>
<td>0.02282</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61.2170</td>
<td>0.1182</td>
<td>-0.2389</td>
<td>0.02041</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-7.9314</td>
<td>0.08357</td>
<td></td>
<td></td>
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<table>
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<tr>
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<th><strong>Uniform Distribution</strong></th>
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<tr>
<td>IM</td>
<td>62.7793</td>
<td>0.08264</td>
<td>0.1995</td>
<td>0.02297</td>
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</tr>
<tr>
<td></td>
<td>61.2166</td>
<td>0.1169</td>
<td>-0.2389</td>
<td>0.02054</td>
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</tr>
<tr>
<td></td>
<td>-7.9314</td>
<td>0.08264</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>62.7800</td>
<td>0.08285</td>
<td>0.1996</td>
<td>0.02300</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61.2160</td>
<td>0.1172</td>
<td>-0.2389</td>
<td>0.02057</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-7.9310</td>
<td>0.08285</td>
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<th><strong>Logistic Distribution</strong></th>
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<tbody>
<tr>
<td>IM</td>
<td>62.7796</td>
<td>0.08296</td>
<td>0.1996</td>
<td>0.02406</td>
<td></td>
</tr>
<tr>
<td></td>
<td>61.2163</td>
<td>0.1173</td>
<td>-0.2390</td>
<td>0.02152</td>
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<td></td>
<td>-7.9311</td>
<td>0.08296</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>SM</td>
<td>62.7790</td>
<td>0.08357</td>
<td>0.1995</td>
<td>0.02282</td>
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<tr>
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<td>61.2170</td>
<td>0.1182</td>
<td>-0.2389</td>
<td>0.02041</td>
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<td></td>
<td>-7.9314</td>
<td>0.08357</td>
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<th><strong>Gamma Distribution</strong></th>
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<tbody>
<tr>
<td>IM</td>
<td>62.7799</td>
<td>0.08213</td>
<td>0.1996</td>
<td>0.02273</td>
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</tr>
<tr>
<td></td>
<td>61.2159</td>
<td>0.1162</td>
<td>-0.2389</td>
<td>0.02033</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-7.9308</td>
<td>0.08213</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td>62.7800</td>
<td>0.05235</td>
<td>0.1983</td>
<td>0.01441</td>
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<tr>
<td></td>
<td>61.2160</td>
<td>0.07403</td>
<td>-0.2378</td>
<td>0.01288</td>
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<td></td>
<td>-7.9306</td>
<td>0.05235</td>
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<table>
<thead>
<tr>
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<th><strong>Exponential Distribution</strong></th>
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<tbody>
<tr>
<td>IM</td>
<td>62.7700</td>
<td>0.8305</td>
<td>2.2251</td>
<td>36.0722</td>
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<td></td>
<td>61.2298</td>
<td>1.1746</td>
<td>-2.0501</td>
<td>32.2549</td>
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<td></td>
<td>-7.9407</td>
<td>0.8305</td>
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<tr>
<td>SM</td>
<td>62.7650</td>
<td>0.8503</td>
<td>2.1457</td>
<td>37.1980</td>
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</tr>
<tr>
<td></td>
<td>61.2370</td>
<td>1.2025</td>
<td>-1.9791</td>
<td>33.2620</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-7.9456</td>
<td>0.8503</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.2.5 Comparison of stochastic sensitivity between PM and MCS

In the table 5.19, the response sensitivities with respect to three bar areas are calculated by the perturbation method of stochastic sensitivity and simulated by MCS with 19,500, 14,300 and 14,450 realizations, respectively. It also be noted that the agreement of expected value of sensitivity is better than that of standard deviation. The agreement of force sensitivity is the best one among three response sensitivities. The displacement sensitivity is at the second place. The accuracy of the second-order approximation in the perturbation method is better than that of the first-order approximation.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Responses</th>
<th>Perturbation Method</th>
<th>MCS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\mu^I$</td>
<td>$\mu^II$</td>
</tr>
<tr>
<td>F1</td>
<td>A1</td>
<td>21.5428</td>
<td>21.7763</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>-0.6803</td>
<td>-0.6834</td>
</tr>
<tr>
<td></td>
<td>A4</td>
<td>-30.4661</td>
<td>-30.7964</td>
</tr>
<tr>
<td>F2</td>
<td>A1</td>
<td>29.7066</td>
<td>29.9804</td>
</tr>
<tr>
<td></td>
<td>A2</td>
<td>0.9621</td>
<td>0.9664</td>
</tr>
<tr>
<td></td>
<td>A2</td>
<td>29.7066</td>
<td>29.9804</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>0.9621</td>
<td>0.9664</td>
</tr>
<tr>
<td>X1</td>
<td>A1</td>
<td>-0.1734</td>
<td>-0.1788</td>
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<tr>
<td></td>
<td>A2</td>
<td>-0.03501</td>
<td>-0.03671</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>-0.007742</td>
<td>-0.007914</td>
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<tr>
<td>X2</td>
<td>A1</td>
<td>0.1016</td>
<td>0.1052</td>
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<tr>
<td></td>
<td>A2</td>
<td>0.1050</td>
<td>0.1083</td>
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<tr>
<td></td>
<td>A3</td>
<td>-0.003207</td>
<td>-0.003226</td>
</tr>
<tr>
<td>S1</td>
<td>A1</td>
<td>-41.2375</td>
<td>-42.1726</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>-0.6803</td>
<td>-0.6971</td>
</tr>
<tr>
<td></td>
<td>A2</td>
<td>-31.5086</td>
<td>-32.1808</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>0.9621</td>
<td>0.9593</td>
</tr>
<tr>
<td>S3</td>
<td>A1</td>
<td>10.7714</td>
<td>10.9267</td>
</tr>
<tr>
<td></td>
<td>A2</td>
<td>-10.5029</td>
<td>-10.6361</td>
</tr>
<tr>
<td></td>
<td>A3</td>
<td>1.6424</td>
<td>1.6564</td>
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</tbody>
</table>
5.2.6 Minimum standard deviation of weight versus the probability of occurrence $p$

In optimization of standard deviation of weight, it is assumed that the material densities have the mean, standard deviation and correlation coefficient matrix as follows:

$$
\begin{bmatrix}
\mu_{\rho_1} \\
\mu_{\rho_2} \\
\mu_{\rho_3}
\end{bmatrix} =
\begin{bmatrix}
0.289 \\
0.289 \\
0.289
\end{bmatrix}
\text{lb} f / \text{in}^3

\begin{bmatrix}
\sigma_{\rho_1} \\
\sigma_{\rho_2} \\
\sigma_{\rho_3}
\end{bmatrix} =
\begin{bmatrix}
0.0289 \\
0.0289 \\
0.0204
\end{bmatrix}
\text{lb} f / \text{in}^3

\begin{bmatrix}
\rho_{AA} =
\begin{bmatrix}
1.00 & 0.502 & 0.354 \\
0.502 & 1.00 & 0.569 \\
0.354 & 0.569 & 1.00
\end{bmatrix}
\end{bmatrix}.
$$

Other allowable values are the same as that in Example 10. The optimal results for standard deviation of objective function are shown in Fig. 5.72. It should be noted that the optimal standard deviation of weight is increased with increasing probability of success like that of the optimal expected weight. The optimization results for both the first- and second-order approximations in the stochastic analysis have little differences, since the variances are small.

![Fig. 5.72 Optimal standard deviation of weight versus the probability of occurrence $p$.](image-url)
5.2.7 Robust design optimization in Example 10

In the robust optimal design, $\mu_w^*$ and $\sigma_w^*$ are the optimal value of weights considering only the mean and the standard deviation, respectively. The constraints are the same as that of minimum expected weight. The optimal solutions corresponding to different weighting factors are listed in Table 5.20, by using the second-order approximation in the perturbation method. Compared with the minimum mean value and standard deviation at $\alpha = 0$, the optimal mean values and standard deviations of other weighting factor change a little, and the design variable $A_1$ increases 0.002 in. For first- and second-order approximation of the perturbation method, the optimal solutions corresponding to different weighting factors are shown in Fig. 5.73 and 5.74, respectively. It may be noted that the expected value increases and the standard deviation decreases as the weighting factor increases, but both variations are small. The conflict between two objectives reveals the characteristic of Pareto solutions.

### Table 5.20 Optimal solution of three-bar truss by using the 2nd-order approximation.

<table>
<thead>
<tr>
<th></th>
<th>Initial</th>
<th>$\alpha = 0.00$</th>
<th>$\alpha = 0.25$</th>
<th>$\alpha = 0.50$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.00$</th>
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</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1.0</td>
<td>1.1042</td>
<td>1.1042</td>
<td>1.1042</td>
<td>1.1042</td>
<td>1.1042</td>
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<tr>
<td>$A_2$</td>
<td>1.0</td>
<td>0.7359</td>
<td>0.7359</td>
<td>0.7360</td>
<td>0.7360</td>
<td>0.7361</td>
</tr>
<tr>
<td>$A_3$</td>
<td>2.0</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
<td>0.1000</td>
</tr>
<tr>
<td>$\mu_w$</td>
<td>151.5123</td>
<td>70.4863</td>
<td>70.4863</td>
<td>70.4870</td>
<td>70.4889</td>
<td>70.4908</td>
</tr>
<tr>
<td>$\sigma_w$</td>
<td>13.1799</td>
<td>8.4499</td>
<td>8.4499</td>
<td>8.4500</td>
<td>8.4502</td>
<td>8.4504</td>
</tr>
<tr>
<td>$f_w$</td>
<td>0.5800 ((\alpha = 0.50))</td>
<td>0.3161</td>
<td>0.3141</td>
<td>0.3121</td>
<td>0.3101</td>
<td>0.3081</td>
</tr>
</tbody>
</table>
5.73 Optimal mean and standard deviation of weight versus weighting factor $a$ by using the first-order approximation.

![Graph showing mean and standard deviation of weight versus weighting factor $a$.](image1)

Fig. 5.74 Optimal mean and standard deviation of weight versus weighting factor $a$ by using the second-order approximation.

![Graph showing mean and standard deviation of weight versus weighting factor $a$.](image2)
CHAPTER VI
CONCLUSIONS AND FUTURE WORK

Within the framework of linear static structural analysis, analytical formulas of stochastic analysis for Integrated Force Method and Dual Integrated Force Method (IFM/IFMD), including the first- and second-order perturbation, have been developed in the present study. According to Neumann series, an efficient statistic approach, Neumann expansion with Monte Carlo simulation, has also been explored for stochastic analysis of IFM/IFMD. Moreover, stochastic variational principles of IFM have been investigated in theory. In the same time, simplification of stochastic analysis on computation is considered, too.

In stochastic sensitivity analysis, three methods (the perturbation method, Neumann expansion with Monte Carlo simulation and the reliability-based method) have been applied to the response sensitivity analysis.

Based on stochastic analysis of IFM/IFMD, the merit function and stochastic constraints are formulated, both minimum expectation of objective function and robust design are performed well on CometBoards using sequential quadratic programming method.

The perturbation method and Neumann expansion technique for stochastic analysis of IFM/IFMD have been programmed in FORTRAN and Maple V codes. Their
analytical and optimal solutions of 15 structures have been obtained. More comparisons and simulations for three-bar truss are investigated.

Based on these numerical studies, we arrived at the following conclusions:

1) For the stochastic analysis of IFM/IFMD, the perturbation method is feasible and efficient under the relatively small fluctuation in random variables about their mean. The first- and second-order approximations have very small differences on stochastic responses for the small covariance matrix of the primitive random variables. The stochastic analysis can be easily incorporated into IFM/IFMD programs.

2) In sensitivity analysis, the force response is most sensitive to the mechanical load and least sensitive to Young’s modulus. The displacement response is most sensitive to the Young’s modulus and the mechanical load component. The stress response is most sensitive to the mechanical load component and the corresponding area, or moment of inertia and least sensitive to Young’s modulus. The deterministic sensitivity has small difference with the sensitivity of two stochastic approximations. The difference of sensitivities of two stochastic approximations is very small, even equal under small covariance matrix of the primitive random variables.

3) The optimal expected weight increases/reduces with increasing/reducing the probability of success. The minimum standard deviation of weight has the same pattern as the expected weight. The difference between two approximations in the stochastic analysis is small for the optimal expectation and standard deviation of weight.
4) For multi random variables, the second-order perturbation analysis shows better agreement with Monte Carlo simulation results than the first-order one. The mean value and standard deviation of displacements have a fairly good agreement with MCS obtained from IFM and SM.

5) Time saving depends on the order of expansion in Neumann expansion with Monte Carlo simulation. The convergence rate of the expected value is faster than that of the standard deviation. Results indicate that for coefficient of variance of 0.1, the first- and second-order perturbation results are in good agreement with direct simulation results from SM, the standard deviation of force is in good agreement with that of the direct simulation of IFM. However, the results are problem-dependent.

6) The perturbation method is applicable to the stochastic analysis with random Young’s modulus for normal, uniform, and logistic distributions. Other distributions need more investigation.

7) The sensitivity of the second-order perturbation method turns out to be quite accurate about mean values of response sensitivity, comparing to Monte Carlo simulation results.

8) The robust design solution demonstrates that reducing standard deviation of the objective function often causes an increase of its expected value. A better design may be chosen from the set of Pareto solutions with different weighting factors.

Due to limited time, the stochastic analysis and optimization in linear static structure are investigated in the present study, by using IFM/IFMD. In future, a wider range of stochastic analysis, including buckling, free vibration and dynamic analysis, can
be dealt with the perturbation method based on the finite element method and can be included in design optimization with CometBoards.
REFERENCES


[98] Mckay, M. D. Conover, W. J. and Beckman, R. J., A comparion of three methods for selecting values of input variables in the analysis of output from a computer code, Technometrics, vol. 21(2), 1979, pp. 239-245.


[103] Sten, M., Large sample proporties of simulations using Latin hypercube sampling, Technometrics, 29(2) 1987, pp. 143-51.


APPENDIX

1. The follow chart of the stochastic analysis and optimization

Here follow chart of the stochastic analysis and optimization is shown as following:

![Stochastic Analysis Diagram]

- **INPUT**
  - **CALCULATING**:
    - 1. TOTAL LOAD
    - 2. TOTAL DEFORMATIONS
  - **CALCULATING**:
    - 1. AREA MATRIX
    - 2. YOUNG'S MODULUS MATRIX
  - **CALCULATING**:
    - 1. FLEXIBILITY MATRIX
    - 2. GOVERNING MATRIX
  - **CALCULATING**: FORCE VECTOR:
    - 1. DETERMINISTIC VALUE
    - 2. MEAN VALUE
    - 3. CONVARIANCE MATRIX
  - **CALCULATING**: DISPLACEMENT VECTOR:
    - 1. DETERMINISTIC VALUE
    - 2. MEAN VALUE
    - 3. CONVARIANCE MATRIX
Fig A. Flow chart of the stochastic analysis and optimization for IFM

2. The Fortran code of the stochastic analysis and optimization for example 10

********************************************************************************************************************
* This Program is to evaluate the Mean values and Covariance matrices for Stochastic Analysis of the Three-bar
* Input: (1) Mean Value and Covariance Matrice of Random Variables such as Bar Areas A[i], Young's Modulus E, Coefficient of Thermal Expansion alpha[i], Mechanical Loads P[i], Temperature T[i] and Settling Support Delta[i].
* (2) Equilibrium Matrix [B] and Compatibility Matrix [C]
* (3) Other necessary parameters
* Output: (1) Mean Values and Covariance Matrix of Internal Forces
* (2) Mean Values and Covariance Matrix of Displacements
* (3) Mean Values and Covariance Matrix of Stresses
* (4) Mean Values and Covariance Matrix of Strains
* The last Part includes calculations for the three stochastic behavior
********************************************************************************************************************
* constraints: \( P(g(j) \leq g(uj)) \geq p \)

* Input: the specified percent probability, \( p \) (0 <= \( p \) <= 1.0)

* the allowable strength, \( \text{Strength}_m, \text{Strength}_d \)

* Output: 
  \( g[1] = (\text{abs}(\text{Sigma}\_\text{Mean}[1,1]/\text{Strength}_m) - 1) + \Phi(p)\sqrt{\text{Cov}\_\text{Sigma}[1,1]+\text{Strength}_d^2/\text{Strength}_m} \)
  \( g[2] = (\text{abs}(\text{Sigma}\_\text{Mean}[2,1]/\text{Strength}_m) - 1) + \Phi(p)\sqrt{\text{Cov}\_\text{Sigma}[2,2]+\text{Strength}_d^2/\text{Strength}_m} \)
  \( g[3] = (\text{abs}(\text{Sigma}\_\text{Mean}[3,1]/\text{Strength}_m) - 1) + \Phi(p)\sqrt{\text{Cov}\_\text{Sigma}[3,3]+\text{Strength}_d^2/\text{Strength}_m} \)
  \( g[4] = (\text{abs}(\text{X}\_\text{Mean}[3,1]/\text{X}_\text{m}) - 1) + \Phi(p)\sqrt{\text{Cov}\_\text{X}[3,3]+\text{X}_d^2/\text{X}_m} \)
  \( g[5] = (\text{abs}(\text{Sigma}\_\text{Mean}[3,1]/\text{Strength}_m) - 1) + \Phi(p)\sqrt{\text{Cov}\_\text{Sigma}[3,3]+\text{Strength}_d^2/\text{Strength}_m} \)

********************************************************************************

PROGRAM MAIN
C USE NUMERICAL_LIBRARIES
C USE IMSLF90
PARAMETER(M=2,N=3,NN=10)
DIMENSION IS(N), JS(N)
DOUBLE PRECISION B(M,N), C(N-M,N), Br(N,2)
DOUBLE PRECISION LENGTH(N), AREA(N), DEPTH(N), P(2), DELTA(2)
DOUBLE PRECISION RHO(NN,NN)
DOUBLE PRECISION E, ALPHA, TEMP, CP_TIME
C *******************************************************************
DOUBLE PRECISION BETA_M_BAR(N,1),BETA_T_BAR(N,1),BETA_S_BAR(N,1),
*                                              BETA_BAR(N,1), \( \Phi(p)\sqrt{\text{Cov}\_\text{Sigma}[3,3]+\text{Strength}_d^2/\text{Strength}_m} \)
* GAMMA4(N,N), GAMMA5(N,N), GAMMA6(N,N),
* GAMMA7(N,N), GAMMA8(N,N)
DOUBLE PRECISION DELTA_F(N,1), DELTA_X(M,1), DELTA_SIGMA(N,1),
* DELTA_EPSILON(N,1)
DOUBLE PRECISION RATIO_F(N,1), RATIO_X(M,1), RATIO_SIGMA(N,1),
* RATIO_EPSILON(N,1)
DOUBLE PRECISION STD_DER_F(N,1), STD_DER_X(M,1),
* STD_DER_SIGMA(N,1), STD_DER_EPSILON(N,1)
DOUBLE PRECISION RHO_F(N,N), RHO_X(M,M),
* RHO_SIGMA(N,N), RHO_EPSILON(N,N)
DOUBLE PRECISION COV_F_PER(N,N), COV_X_PER(M,M),
* COV_SIGMA_PER(N,N), COV_EPSILON_PER(N,N)
DOUBLE PRECISION Y_F, Y_X, Y_SIGMA, Y_EPSILON
REAL PERCENT
DOUBLE PRECISION STRENGTH_M, STRENGTH_D, X_M, X_D
DOUBLE PRECISION PHI_P, G_SIGMA_CONSTRAINT(N), G_X_CONSTRAINT(M)
DOUBLE PRECISION DENSITY, THICKNESS, WEIGHT

C The initial data for three-bar truss
DATA B/ 0.7071067812D0, -0.7071067812D0, 0.0,
* -1.D0, -0.7071067812D0, -0.7071067812D0/
DATA C/ 0.7071067812D0, -1.0, 0.7071067812D0/
DATA Br/-0.7071067812D0, 0.D0, 0.D0,
* 0.7071067812D0, 0.D0, 0.D0/
DATA LENGTH/141.4213562373D0, 100.D0, 141.4213562373D0/
DATA AREA/1.0,1.0,2.0/
DATA TEMP/1.D02/
DATA E/3.0E07/
DATA ALPHA/6.6D-06/
DATA P/5.D04,-1.D05/
DATA DELTA/1.D-01, 1.5D-01/
DATA DEPTH/0.D0, 0.D0, 0.D0/
DATA RHO/ 1.D-02, 5.D-03, 1.25D-03, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0,
* 5.D-03, 1.D-02, 1.25D-03, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0,
* 1.25D-03, 1.25D-3, 2.5D-03, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 1.D-02, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 1.D-02, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 1.D-02,
* 1.25D-03, 0.D0, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 1.25D-3,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 2.5D-03, 2.5D-03,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 2.5D-03, 2.5D-03,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 0.D0,
* 0.D0, 0.D0, 0.D0, 0.D0, 0.D0, 2.5D-03, 1.D-02/

C The following datum are the specified percent probability, and the mean value
C and standard derivations of the allowable strength

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DATA PERCENT/0.75/
DATA STRENGTH_M,STRENGTH_D/2.D04,2.D03/
DATA X_M,X_D/0.20,0.015/
DATA DENSITY/0.289/
C Read in the areas on standard in...

READ(*,*) AREA

C Calculating mechanical loads [Pm_bar]
P_M_BAR(1,1) = P(1)
P_M_BAR(2,1) = P(2)
P_M_BAR(3,1) = 0.D0
C Calculating uniform loads [Pm_beta_bar]
BETA_M_BAR(1,1) = 0.D0
BETA_M_BAR(2,1) = 0.D0
BETA_M_BAR(3,1) = 0.D0
R = MATMUL(C,BETA_M_BAR)
P_MB_BAR(1,1) = 0.D0
P_MB_BAR(2,1) = 0.D0
P_MB_BAR(3,1) = -R(1,1)
C Calculating thermal loads [Pt_bar]
DO 10 I = 1,N
   BETA_T_BAR(I,1) = ALPHA*TEMP*LENGTH(I)/2.0
10 CONTINUE
R = MATMUL(C,BETA_T_BAR)
P_T_BAR(1,1) = 0.D0
P_T_BAR(2,1) = 0.D0
P_T_BAR(3,1) = -R(1,1)
C Calculating settling support loads [Ps_bar]
X(1,1) = DELTA(1)
X(2,1) = DELTA(2)
BETA_S_BAR = -MATMUL(Br,X)
R = MATMUL(C,BETA_S_BAR)
P_S_BAR(1,1) = 0.D0
P_S_BAR(2,1) = 0.D0
P_S_BAR(3,1) = -R(1,1)
C Calculating total loads [P_bar] and total deformation [beta_bar]
BETA_BAR = BETA_M_BAR + BETA_T_BAR + BETA_S_BAR
P_BAR = P_M_BAR + P_MB_BAR + P_T_BAR + P_S_BAR
C Forming the area matrix [A_bar]
DO 20 I = 1,N
   DO 20 J = 1,N
      IF (I.EQ.J) THEN
         AREA_BAR(I,J) = AREA(J)
      ELSE
         AREA_BAR(I,J) = 0.D0
      ENDIF
20 CONTINUE
DO 25 I = 1,N
   DO 25 J = 1,N
      AREA_INV_BAR(I,J) = AREA_BAR(I,J)
25 CONTINUE
CALL MAT_INV(AREA_INV_BAR,N,L,IS,JS)
C Forming the Young's modulus matrix [E_bar]
DO 30 I = 1,N
DO 30 J = 1,N
IF (I.EQ.J) THEN
   E_BAR(I,J) = E
ELSE
   E_BAR(I,J) = 0.D0
ENDIF
30 CONTINUE
DO 35 I = 1,N
DO 35 J = 1,N
   E_INV_BAR(I,J) = E_BAR(I,J)
35 CONTINUE
CALL MAT_INV(E_INV_BAR,N,L,IS,JS)
C Calculating the flexibility matrix \(G_{\text{bar}}\) and \(G_{\text{inv bar}}\)
DO 40 I = 1,N
DO 40 J = 1,N
IF (I .EQ. J) THEN
   G_BAR(I,I) = LENGTH(I)/E/AREA(I)
ELSE
   G_BAR(I,J) = 0.D0
ENDIF
40 CONTINUE
C Calculating \([CG]\)
CG = MATMUL(C,G_BAR)
C Forming the governing matrix \([S_{\text{bar}}]\) and \([S_{\text{inv bar}}]\)
DO 50 I = 1, N
DO 50 J = 1, N
IF (I.LE.M) THEN
   S_BAR(I,J) = B(I,J)
ELSE
   S_BAR(I,J) = CG(I-M,J)
ENDIF
50 CONTINUE
DO 55 I = 1,N
DO 55 J = 1,N
   S_INV_BAR(I,J) = S_BAR(I,J)
55 CONTINUE
CALL MAT_INV(S_INV_BAR,N,L,IS,JS)
ST = TRANPOSE(S_INV_BAR)
C Forming the deformation coefficient matrix \([J]\)
DO 60 I = 1,M
DO 60 J = 1,N
   JR(I,J) = ST(I,J)
60 CONTINUE
C Calculating the deterministic value of force vector \(\{F_{\text{bar}}\}\)
F_BAR = MATMUL(S_INV_BAR,P_BAR)
C Calculating the mean value of force vector \(\{F_{\text{mean}}\}\)
PHI_F = 0.D0
PSI_F = 0.D0
DO 70 I = 1,NN
DO 70 J = 1,NN
   CALL S_FIRST_DER(I,N,S_BAR,S_FI)
   CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
   CALL S_SECOND_DER(I,J,N,S_BAR,S_SIJ)
   CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
CALL P_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,P,P_FJ)
PHI_F = PHI_F + RHO(I,J)*(2.0*MATMUL(MATMUL(S_FI,S_INV_BAR),S_FJ)-S_SIJ)
PSI_F = PSI_F + RHO(I,J)*(2.0*MATMUL(MATMUL(S_FI,S_INV_BAR),P_FJ)-P_SIJ)
70 CONTINUE
F_MEAN = F_BAR+0.5D0*(MATMUL(S_INV_BAR,MATMUL(PHI_F,F_BAR)-PSI_F))

C Calculating the covariance matrix of force [cov_F]
LAMBDA1 = 0.D0
LAMBDA2 = 0.D0
LAMBDA3 = 0.D0
LAMBDA4 = 0.D0
DO 80 I = 1,NN
DO 80 J = 1,NN
CALL S_FIRST_DER(I,N,S_BAR,S_FI)
CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
CALL P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,P,P_FI)
CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,P,P_FJ)
LAMBDA1 = LAMBDA1 + RHO(I,J)*(MATMUL(S_FI,MATMUL(F_BAR,MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(S_FJ)))))
LAMBDA2 = LAMBDA2 + RHO(I,J)*(MATMUL(S_FI,MATMUL(F_BAR,TRANSPOSE(P_FJ)))))
LAMBDA3 = LAMBDA3 + RHO(I,J)*(MATMUL(P_FI,MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(S_FJ)))))
LAMBDA4 = LAMBDA4 + RHO(I,J)*MATMUL(P_FI,TRANSPOSE(P_FJ))
80 CONTINUE
COV_F = MATMUL(S_INV_BAR,MATMUL((LAMBDA1-LAMBDA2-LAMBDA3+LAMBDA4)
                 ,TRANSPOSE(S_INV_BAR)))

C Calculating the deterministic value of displacement vector \{X_bar\}
X_BAR = MATMUL(JR,(MATMUL(G_BAR,F_BAR) + BETA_BAR))

C Calculating the mean value of displacement vector \{X_mean\}
PHI_X = 0.D0
PSI_X = 0.D0
DO 90 I = 1,NN
DO 90 J = 1,NN
CALL G_FIRST_DER(I,N,G_BAR,G_FI)
CALL G_SECOND_DER(I,J,N,G_BAR,G_SIJ)
CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,P,P_FJ)
CALL BETA_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,P,BETA_SIJ)
PHI_X = PHI_X + RHO(I,J)*(2.0*MATMUL(MATMUL(G_FI,S_INV_BAR),S_FJ)-G_SIJ)
PSI_X = PSI_X + RHO(I,J)*(2.0*MATMUL(MATMUL(G_FI,S_INV_BAR),P_FJ)+BETA_SIJ)
90 CONTINUE
X_MEAN = X_BAR + 0.5D0*MATMUL(JR,(MATMUL(MATMUL(G_BAR,S_INV_BAR),
                 MATRIX))^T))
Calculating the covariance matrix of displacement [cov_X]

LAMBDA5 = 0.D0
LAMBDA6 = 0.D0
LAMBDA7 = 0.D0
LAMBDA8 = 0.D0
LAMBDA9 = 0.D0
LAMBDA10 = 0.D0
LAMBDA11 = 0.D0
LAMBDA12 = 0.D0
LAMBDA13 = 0.D0
LAMBDA14 = 0.D0
LAMBDA15 = 0.D0
LAMBDA16 = 0.D0
DO 100 I = 1,NN
  DO 100 J = 1,NN
    CALL S_FIRST_DER(I,N,S_BAR,S_FI)
    CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
    CALL G_FIRST_DER(I,N,G_BAR,G_FI)
    CALL G_FIRST_DER(J,N,G_BAR,G_FJ)
    CALL P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
      P,P_FI)
    CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
      P,P_FJ)
    CALL BETA_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
      AREA,E,P,BETA_FI)
    CALL BETA_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
      AREA,E,P,BETA_FJ)
    LAMBDA5 = LAMBDA5 + RHO(I,J)*(MATMUL(G_FI,MATMUL(F_BAR,
      MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(G_FJ))))
    LAMBDA6 = LAMBDA6 + RHO(I,J)*(MATMUL(G_FI,MATMUL(F_BAR,
      TRANSPOSE(BETA_FJ)))
    LAMBDA7 = LAMBDA7 + RHO(I,J)*(MATMUL(BETA_FI,MATMUL(
      TRANSPOSE(F_BAR),TRANSPOSE(G_FJ)))
    LAMBDA8 = LAMBDA8 + RHO(I,J)*MATMUL(BETA_FI,
      TRANSPOSE(BETA_FJ))
    LAMBDA9 = LAMBDA9 + RHO(I,J)*(MATMUL(S_FI,MATMUL(F_BAR,
      MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(G_FJ))))
    LAMBDA10 = LAMBDA10 + RHO(I,J)*(MATMUL(S_FI,MATMUL(F_BAR,
      TRANSPOSE(BETA_FJ)))
    LAMBDA11 = LAMBDA11 + RHO(I,J)*(MATMUL(P_FI,MATMUL(
      TRANSPOSE(F_BAR),TRANSPOSE(G_FJ)))
    LAMBDA12 = LAMBDA12 + RHO(I,J)*MATMUL(P_FI,
      TRANSPOSE(BETA_FJ))
    LAMBDA13 = LAMBDA13 + RHO(I,J)*(MATMUL(G_FI,MATMUL(F_BAR,
      MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(S_FJ))))
    LAMBDA14 = LAMBDA14 + RHO(I,J)*(MATMUL(G_FI,MATMUL(F_BAR,
      TRANSPOSE(P_FJ)))
    LAMBDA15 = LAMBDA15 + RHO(I,J)*(MATMUL(BETA_FI,MATMUL(
      TRANSPOSE(F_BAR),TRANSPOSE(S_FJ)))
    LAMBDA16 = LAMBDA16 + RHO(I,J)*MATMUL(BETA_FI,
      TRANSPOSE(P_FJ))
  CONTINUE
COV_X = MATMUL(JR,MATMUL((MATMUL(G_BAR,
* MATMUL(COV_F,TRANSPOSE(G_BAR))) + (LAMBDA5 + LAMBDA6 +
* LAMBDA7 + LAMBDA8) + MATMUL(G_BAR, MATMUL(S_INV_BAR,
* (-LAMBDA9 - LAMBDA10 + LAMBDA11 + LAMBDA12)) +
* MATMUL(MATMUL((-LAMBDA13 + LAMBDA14 - LAMBDA15 +
* LAMBDA16),TRANSPOSE(S_INV_BAR)),TRANSPOSE(G_BAR))),
* TRANSPOSE(JR)))

C Calculating the deterministic value of stress vector \{\Sigma_{\text{bar}}\}
SIGMA_BAR = MATMUL(AREA_INV_BAR,F_BAR)

C Calculating the mean value of stress vector \{\Sigma_{\text{mean}}\}
PHI_SIGMA = 0.D0
PSI_SIGMA = 0.D0
DO 110 I = 1,NN
  DO 110 J = 1,NN
    CALL S_FIRST_DER(I,N,S_BAR,S_FI)
    CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
    CALL S_SECOND_DER(I,J,N,S_BAR,S_SIJ)
    CALL AREA_FIRST_DER(I,N,AREA_BAR,AREA_FI)
    CALL AREA_FIRST_DER(J,N,AREA_BAR,AREA_FJ)
    CALL AREA_SECOND_DER(I,J,N,AREA_BAR,AREA_SIJ)
    CALL P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
    * P,P_FI)
    CALL P_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
    * P,P_FJ)
    PHI_SIGMA = PHI_SIGMA + RHO(I,J)*( MATMUL((2.D0*MATMUL(MATMUL(
    * S_FI,S_INV_BAR),S_FJ) - S_SIJ),AREA_BAR) + MATMUL(
    * S_BAR,(2.D0*MATMUL(MATMUL(AREA_FI,AREA_INV_BAR),
    * AREA_FJ) - AREA_SIJ)) + 2.D0*MATMUL(S_BAR,MATMUL(
    * AREA_FI,TRANSPOSE(TRANSPOSE(AREA_BAR)))))
  110 CONTINUE
SIGMA_MEAN = SIGMA_BAR + 0.5D0*(MATMUL(AREA_INV_BAR,MATMUL(
* S_INV_BAR,(MATMUL(PHI_SIGMA,SIGMA_BAR) - PSI_SIGMA))))

C Calculating the covariance matrix of stress [cov_\Sigma]
GAMMA1 = 0.D0
GAMMA2 = 0.D0
GAMMA3 = 0.D0
DO 120 I = 1,NN
  DO 120 J = 1,NN
    CALL S_FIRST_DER(I,N,S_BAR,S_FI)
    CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
    CALL AREA_FIRST_DER(I,N,AREA_BAR,AREA_FI)
    CALL AREA_FIRST_DER(J,N,AREA_BAR,AREA_FJ)
    CALL P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
    * P,P_FI)
    CALL P_FIRST_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
    * P,P_FJ)
    GAMMA1 = GAMMA1 + RHO(I,J)*(MATMUL(TRANSPOSE(SIGMA_BAR),TRANSPOSE(S_INV_BAR))),
    * MATMUL(TRANSPOSE(SIGMA_BAR),TRANSPOSE(AREA_FJ))))
GAMMA2 = GAMMA2 +RHO(I,J)*(MATMUL(TRANSPOSE(S_INV_BAR),TRANSPOSE(G_BAR))))
* MATMUL((MATMUL(TRANSPOSE(F_BAR),TRANSPOSE(S_FJ))
*   - TRANSPOSE(P_FJ)),TRANSPOSE(S_INV_BAR))))
  GAMMA3 = GAMMA3 + RHO(I,J)*(MATMUL(S_INV_BAR,MATMUL((
  * MATMUL(S_FI,F_BAR) - P_FI),MATMUL(TRANSPOSE(
  * SIGMA_BAR),TRANSPOSE(AREA_FJ)))))
120 CONTINUE
  COV_SIGMA = MATMUL(ARENA_INV_BAR,MATMUL((COV_F + (GAMMA1 + GAMMA2
  * + GAMMA3)),TRANSPOSE(AREA_INV_BAR)))

C Calculating the deterministic value of strain vector \{Epsilon_bar\}

EPSILON_BAR = MATMUL(E_INV_BAR,SIGMA_BAR)

C Calculating the mean value of strain vector \{Epsilon_mean\}

PHI_EPSILON = 0.D0
PSI_EPSILON = 0.D0
DO 130 I = 1,NN
  DO 130 J = 1,NN
    CALL S_FIRST_DER(I,N,S_BAR,S_FI)
    CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
    CALL S_SECOND_DER(I,J,N,S_BAR,S_SIJ)
    CALL AREA_FIRST_DER(I,N,AREA_BAR,AREA_FI)
    CALL AREA_FIRST_DER(J,N,AREA_BAR,AREA_FJ)
    CALL AREA_SECOND_DER(I,J,N,AREA_BAR,AREA_SIJ)
    CALL E_FIRST_DER(I,N,E_BAR,E_FI)
    CALL E_FIRST_DER(J,N,E_BAR,E_FJ)
    CALL E_SECOND_DER(I,J,N,E_BAR,E_SIJ)
    CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
    * P,P_FJ)
    CALL P_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
    * AREA,E,P,P_SIJ)
    PHI_EPSILON = PHI_EPSILON + RHO(I,J)*(MATMUL((2.D0*MATMUL(
    * MATMUL(S_FI,S_INV_BAR),S_FJ) - S_SIJ),AREA_BAR),
    * E_BAR) + MATMUL(S_BAR,MATMUL((2.D0*MATMUL(MATMUL(
    * AREA_FI,AREA_INV_BAR),AREA_FJ) - AREA_SIJ,E_BAR)),
    * MATMUL(S_BAR,MATMUL(AREA_BAR,(2.D0*MATMUL(MATMUL(
    * E_FI,E_INV_BAR),E_FJ) - E_SIJ))) + 2.D0*MATMUL(
    * S_BAR,MATMUL(MATMUL(2.D0*MATMUL(MATMUL(
    * E_INV_BAR,MATMUL(S_FJ,MATMUL(AREA_BAR,E_BAR)))) +
    * S_INV_BAR,MATMUL(S_BAR,MATMUL(2.D0*MATMUL(MATMUL(
    * S_INV_BAR,MATMUL(S_FJ,MATMUL(AREA_BAR,E_BAR)))) +
    * S_INV_BAR,MATMUL(S_BAR,MATMUL(2.D0*MATMUL(MATMUL(
    * S_INV_BAR,MATMUL(S_FJ,MATMUL(AREA_BAR,E_BAR)))))))))
    PSI_EPSILON = PSI_EPSILON + RHO(I,J)*((2.D0*MATMUL(MATMUL(S_FI,
    * S_INV_BAR),P_FJ) - P_SIJ) + 2.D0*MATMUL(S_BAR,MATMUL(
    * ARENA_FI,MATMUL(2.D0*MATMUL(MATMUL(S_INV_BAR,P_FJ)
    * S_INV_BAR,MATMUL(S_BAR,MATMUL(AREA_INV_BAR,2.D0*MATMUL(
    * E_FI,MATMUL(2.D0*MATMUL(2.D0*MATMUL(MATMUL(
    * S_INV_BAR,MATMUL(S_FJ,MATMUL(AREA_BAR,E_BAR)))))))))))
  130 CONTINUE
  EPSILON_MEAN = EPSILON_BAR + 0.5D0*(MATMUL(E_INV_BAR,
  * MATMUL(ARENA_INV_BAR,MATMUL(S_INV_BAR,
  * (MATMUL(PHI_EPSILON,EPSILON_BAR) - PSI_EPSILON))))

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C Calculating the covariance matrix of strain $[\text{cov}_\text{Epsilon}]$

\[
\begin{align*}
\text{GAMMA4} &= 0.D0 \\
\text{GAMMA5} &= 0.D0 \\
\text{GAMMA6} &= 0.D0 \\
\text{GAMMA7} &= 0.D0 \\
\text{GAMMA8} &= 0.D0 \\
\end{align*}
\]

DO 140 I = 1,NN
  DO 140 J = 1,NN
    CALL S_FIRST_DER(I,N,S_BAR,S_FI)
    CALL S_FIRST_DER(J,N,S_BAR,S_FJ)
    CALL AREA_FIRST_DER(I,N,AREA_BAR,AREA_FI)
    CALL AREA_FIRST_DER(J,N,AREA_BAR,AREA_FJ)
    CALL E_FIRST_DER(I,N,E_BAR,E_FI)
    CALL E_FIRST_DER(J,N,E_BAR,E_FJ)
    CALL P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
      * P,P_FI)
    CALL P_FIRST_DER(J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
      * P,P_FJ)
    \[
    \begin{align*}
    \text{GAMMA4} &= \text{GAMMA4} + \text{RHO}(I,J) \times \text{MATMUL}(E_FI,\text{MATMUL}(EPILEilon_BAR, \\
    & \quad \text{MATMUL(TRANSPOSE(EPILEilon_BAR),TRANSPOSE(E_FJ))))}) \\
    \text{GAMMA5} &= \text{GAMMA5} + \text{RHO}(I,J) \times \text{MATMUL}(\text{AREA_INV_BAR},\text{MATMUL(} \\
    & \quad \text{AREA_FI,\text{MATMUL}(SIGMA_BAR,\text{MATMUL(TRANSPOSE(} \\
    & \quad \text{EPSILON_BAR),TRANSPOSE(E_FJ))))})}) \\
    \text{GAMMA6} &= \text{GAMMA6} + \text{RHO}(I,J) \times \text{MATMUL}(E_FI,\text{MATMUL}(EPILEilon_BAR, \\
    & \quad \text{MATMUL(TRANSPOSE(SIGMA_BAR),\text{MATMUL(TRANSPOSE(} \\
    & \quad \text{AREA_FJ,TRANSPOSE(\text{AREA_INV_BAR))))}))} \\
    \text{GAMMA7} &= \text{GAMMA7} + \text{RHO}(I,J) \times \text{MATMUL}(E_FI,\text{MATMUL}(EPILEilon_BAR, \\
    & \quad \text{MATMUL((\text{MATMUL(TRANSPOSE(\text{F_BAR),TRANSPOSE(\text{S_FJ}) -} \\
    & \quad \text{TRANSPOSE(P_FJ))},\text{MATMUL(TRANSPOSE(\text{S_INV_BAR),} \\
    & \quad \text{AREA_INV_BAR))}))}) \\
    \text{GAMMA8} &= \text{GAMMA8} + \text{RHO}(I,J) \times \text{MATMUL}(\text{AREA_INV_BAR},\text{MATMUL(} \\
    & \quad \text{S_INV_BAR,\text{MATMUL((\text{MATMUL(\text{S_FI,F_BAR) - P_FI),} \\
    & \quad \text{MATMUL(TRANSPOSE(\text{EPSILON_BAR),TRANSPOSE(E_FJ))))})} \end{align*}
\]

140 CONTINUE

\[
\text{COV}_\text{EPSILON} = \text{MATMUL(\text{E_INV_BAR},\text{MATMUL((\text{COV_SIGMA + (GAMMA4 +} \\
  & \quad \text{GAMMA5 + GAMMA6 + GAMMA7 + GAMMA8)),} \\
  & \quad \text{TRANSPOSE(\text{E_INV_BAR}))})})
\]

C Calculating the relative ratio of mean values for $\{F\}, \{X\}, \{\Sigma\}$ and $\{\text{Epsilon}\}$

\[
\begin{align*}
\text{DELTA_F} &= \text{F_MEAN} - \text{F_BAR} \\
\text{DELTA_X} &= \text{X_MEAN} - \text{X_BAR} \\
\text{DELTA_SIGMA} &= \text{SIGMA_MEAN} - \text{SIGMA_BAR} \\
\text{DELTA_EPSILON} &= \text{EPSILON_MEAN} - \text{EPSILON_BAR}
\end{align*}
\]

DO 150 I = 1,N
  IF (\text{F_MEAN(I,1).EQ.0.D0}) THEN
    \text{RATIO_F(I,1) = 0.D0}
  ELSE
    \text{RATIO_F(I,1) = 1.002*DELTA_F(I,1)/F_MEAN(I,1)}
  ENDIF
150 CONTINUE

DO 160 I = 1,N
  IF (\text{SIGMA_MEAN(I,1).EQ.0.D0}) THEN
    \text{RATIO_SIGMA(I,1) = 0.D0}
  ELSE
    \text{RATIO_SIGMA(I,1) = 1.002*DELTA_SIGMA(I,1)/SIGMA_MEAN(I,1)}
  ENDIF
160 CONTINUE
RATIO_SIGMA(I,1) = 1.0D0*DELTA_SIGMA(I,1)/SIGMA_MEAN(I,1)
ENDIF
160 CONTINUE
DO 170 I = 1,N
IF (EPSILON_MEAN(I,1).EQ.0.D0) THEN
  RATIO_SIGMA(I,1) = 0.D0
ELSE
  RATIO_EPSILON(I,1) = 1.0D0*DELTA_EPSILON(I,1)/EPSILON_MEAN(I,1)
ENDIF
170 CONTINUE
DO 180 I = 1,M
IF (X_MEAN(I,1).EQ.0.D0) THEN
  RATIO_X(I,1) = 0.D0
ELSE
  RATIO_X(I,1) = 1.0D0*DELTA_X(I,1)/X_MEAN(I,1)
ENDIF
180 CONTINUE
C     Calculating the covariance matrices for [Rho_F], [Rho_X],[Rho_SIGMA]and [Rho_EPSILON]
DO 190 I =1,N
STD_DER_F(I,1) = SQRT(COV_F(I,I))
STD_DER_SIGMA(I,1) = SQRT(COV_SIGMA(I,I))
STD_DER_EPSILON(I,1) = SQRT(COV_EPSILON(I,I))
190 CONTINUE
DO 200 I = 1,M
STD_DER_X(I,1) = SQRT(COV_X(I,I))
200 CONTINUE
DO 210 I = 1,N
  DO 210 J = 1,N
    IF ((STD_DER_F(I,1).EQ.0).OR.(STD_DER_F(J,1).EQ.0)) THEN
      RHO_F(I,J) = 0.D0
    ELSE
      RHO_F(I,J) = COV_F(I,J)/(STD_DER_F(I,1)*STD_DER_F(J,1))
    END IF
  210 CONTINUE
DO 220 I =1,N
  DO 220 J = 1,N
    IF ((STD_DER_SIGMA(I,1).EQ.0).OR.(STD_DER_SIGMA(J,1).EQ.0)) THEN
      RHO_SIGMA(I,J) = 0.D0
    ELSE
      RHO_SIGMA(I,J) = COV_SIGMA(I,J)/(STD_DER_SIGMA(I,1)*STD_DER_SIGMA(J,1))
    END IF
  220 CONTINUE
  DO 230 I =1,N
    DO 230 J = 1,N
      IF ((STD_DER_EPSILON(I,1).EQ.0).OR.(STD_DER_EPSILON(J,1).EQ.0)) THEN
        RHO_EPSILON(I,J) = 0.D0
      ELSE
        RHO_EPSILON(I,J) = COV_EPSILON(I,J)/(STD_DER_EPSILON(I,1)*STD_DER_EPSILON(J,1))
      END IF
    230 CONTINUE
  220 CONTINUE
DO 240 I = 1,M
DO 240 J = 1,M
   IF ((STD_DER_X(I,1).EQ.0).OR.(STD_DER_X(J,1).EQ.0)) THEN
      RHO_X(I,J) = 0.D0
   ELSE
      RHO_X(I,J) = COV_X(I,J)/(STD_DER_X(I,1)*STD_DER_X(J,1))
   END IF
240  CONTINUE

Y_F = 0.D0
Y_X = 0.D0
Y_SIGMA = 0.D0
Y_EPSILON = 0.D0
DO 250 I = 1,N
   Y_F = Y_F + F_MEAN(I,1)*F_MEAN(I,1)
   Y_SIGMA = Y_SIGMA + SIGMA_MEAN(I,1)*SIGMA_MEAN(I,1)
   Y_EPSILON = Y_EPSILON + EPSILON_MEAN(I,1)*EPSILON_MEAN(I,1)
250 CONTINUE
DO 260 I = 1,M
   Y_X = Y_X + X_MEAN(I,1)**2
260 CONTINUE
DO 270 I = 1,M
   DO 270 J = 1,M
      COV_F_PER(I,J) = 1.0D02*SQRT(ABS(COV_F(I,J))/Y_F)
      COV_SIGMA_PER(I,J) = 1.0D02*SQRT(ABS(COV_SIGMA(I,J))/Y_SIGMA)
      COV_EPSILON_PER(I,J) = 1.0D02*SQRT(ABS(COV_EPSILON(I,J))/Y_EPSILON)
270 CONTINUE
DO 280 I = 1,M
   DO 280 J = 1,M
      COV_X_PER(I,J) = 1.0D02*SQRT(ABS(COV_X(I,J))/Y_X)
280 CONTINUE

C  Calculating the values of the stochastic behavior constraints of three stresses
C and two displacements
C PHI_P = ANORIN(PERCENT)
PHI_P = 0.6744897502
G_SIGMA_CONSTRAINT(1) = (ABS(SIGMA_MEAN(1,1)/STRENGTH_M) - 1.0) +
   PHI_P*SQRT(COV_SIGMA(1,1)+STRENGTH_D**2)/STRENGTH_M
G_SIGMA_CONSTRAINT(2) = (ABS(SIGMA_MEAN(2,1)/STRENGTH_M) - 1.0) +
   PHI_P*SQRT(COV_SIGMA(2,2)+STRENGTH_D**2)/STRENGTH_M
G_SIGMA_CONSTRAINT(3) = (ABS(SIGMA_MEAN(3,1)/STRENGTH_M) - 1.0) +
   PHI_P*SQRT(COV_SIGMA(3,3)+STRENGTH_D**2)/STRENGTH_M
G_X_CONSTRAINT(1) = (ABS(X_MEAN(1,1)/X_M) - 1.0) +
   PHI_P*SQRT(COV_X(1,1)+X_D**2)/X_M
G_X_CONSTRAINT(2) = (ABS(X_MEAN(2,1)/X_M) - 1.0) +
   PHI_P*SQRT(COV_X(2,2)+X_D**2)/X_M

C  Calculating the mean value of the weight of structure
WEIGHT = DENSITY*(AREA(1)*LENGTH(1) + AREA(2)*LENGTH(2) +
                    AREA(3)*LENGTH(3))

C The computing results for all mean values and covariance matrices
WRITE(*,*)' ',
WRITE(*,*)' **********************************************'
WRITE(*,*) ' *'  
WRITE(*,*) ' Calculating the Mean Values and Covariance'  
WRITE(*,*) ' Matrices of Stochastic Analysis and Stochastic'  
WRITE(*,*) ' Constraints for Example (10)'  
WRITE(*,*) ' *'  
WRITE(*,*) ' ***************************************************'  
WRITE(*,*) '  The Initial Values'  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) '[B]=

WRITE(*,800) N,M  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) '[C]=

WRITE(*,810) ((B(I,J),J=1,N),I=1,M)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'LENGTH

WRITE(*,830) (LENGTH(I),I=1,N)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'DEPTH

WRITE(*,840) (DEPTH(I),I=1,N)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'AREA

WRITE(*,850) (AREA(I),I=1,N)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'TEMP

WRITE(*,860) TEMP  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'E

WRITE(*,870) E  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'ALPHA

WRITE(*,880) ALPHA  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'P[i]

WRITE(*,890) (P(I),I=1,2)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) 'DELTA

WRITE(*,900) (DELTA(I),I=1,2)  
WRITE(*,*) '---------------------------------------------------'  
WRITE(*,*) '[RHO]=

227
WRITE(*,*) '       '
WRITE(*,910) RHO
WRITE(*,*) '---------------------------------------------------'
WRITE(*,*) '[Br]='
WRITE(*,920) Br
WRITE(*,*) '---------------------------------------------------'
WRITE(*,*) '***************************************************
WRITE(*,*) '***************************************************
WRITE(*,*) '       '
WRITE(*,1000) P_M_BAR
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) 'P_T_BAR'
WRITE(*,*) '       '
WRITE(*,1010) P_T_BAR
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) 'P_S_BAR'
WRITE(*,*) '       '
WRITE(*,1020) P_S_BAR
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) 'P_BAR'
WRITE(*,*) '       '
WRITE(*,1030) P_BAR
WRITE(*,*) '==================================================='
WRITE(*,*) 'BETA_T_BAR'
WRITE(*,*) '       '
WRITE(*,1040) BETA_T_BAR
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) 'BETA_S_BAR'
WRITE(*,*) '       '
WRITE(*,1050) BETA_S_BAR
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) 'BETA_BAR'
WRITE(*,*) '       '
WRITE(*,1060) BETA_BAR
WRITE(*,*) '==================================================='
WRITE(*,*) '[AREA_BAR]='
WRITE(*,1070) ((AREA_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) '[AREA_INV_BAR]='
WRITE(*,1080) ((AREA_INV_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) '[E_BAR]='
WRITE(*,1090) (E_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) '[E_INV_BAR]='
WRITE(*,1100) ((E_INV_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*) '---------------------------------------------------
WRITE(*,*) '[G_BAR]='
WRITE(*,1110) (G_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*) '---------------------------------------------------
WRITE(*,*)'  
WRITE(*,1110) ((G_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'[S_BAR]=' 
WRITE(*,*)'  
WRITE(*,1120) ((S_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'[S_INV_BAR]=' 
WRITE(*,*)'  
WRITE(*,1130) ((S_INV_BAR(I,J),J=1,N),I=1,N)
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'[ST]=' 
WRITE(*,*)'  
WRITE(*,1140) ((ST(I,J),J=1,N),I=1,N)
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'       ' 
WRITE(*,*)'       ' 
WRITE(*,*)'***************************************************
WRITE(*,*)'***************************************************
WRITE(*,*)'The Computing Results'
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{F_BAR}=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{F_MEAN}=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{RATIO_F}=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'[COV_F]=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'[COV_F_PER]=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{X_BAR}=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{X_MEAN}=' 
WRITE(*,*)'-----------------------------------------------
WRITE(*,*)'{RATIO_X}='
WRITE(*,*)'-----------------------------------------------
WRITE(*,*) '       '
WRITE(*,1270) RATIO_X
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_X]='
WRITE(*,*)'       '
WRITE(*,1280) ((COV_X(I,J),J=1,M),I=1,M)
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_X_PER]='
WRITE(*,*)'       '
WRITE(*,1290) ((COV_X_PER(I,J),J=1,M),I=1,M)
WRITE(*,*)'{SIGMA_BAR}='
WRITE(*,*)'       '
WRITE(*,1300) SIGMA_BAR
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'{SIGMA_MEAN}='
WRITE(*,*)'       '
WRITE(*,1310) SIGMA_MEAN
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'{RATIO_SIGMA}='
WRITE(*,*)'       '
WRITE(*,1320) RATIO_SIGMA
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_SIGMA]='
WRITE(*,*)'       '
WRITE(*,1330) ((COV_SIGMA(I,J),J=1,N),I=1,N)
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_SIGMA_PER]='
WRITE(*,*)'       '
WRITE(*,1340) ((COV_SIGMA_PER(I,J),J=1,N),I=1,N)
WRITE(*,*)'{EPSILON_BAR}='
WRITE(*,*)'       '
WRITE(*,1350) EPSILON_BAR
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'{EPSILON_MEAN}='
WRITE(*,*)'       '
WRITE(*,1360) EPSILON_MEAN
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'{RATIO_EPSILON}='
WRITE(*,*)'       '
WRITE(*,1370) RATIO_EPSILON
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_EPSILON]='
WRITE(*,*)'       '
WRITE(*,1380) ((COV_EPSILON(I,J),J=1,N),I=1,N)
WRITE(*,*)'---------------------------------------------------'
WRITE(*,*)'[COV_EPSILON_PER]='
WRITE(*,*)'       '
WRITE(*,1390) ((COV_EPSILON_PER(I,J),J=1,N),I=1,N)
WRITE(*,*)'***************************************************
WRITE(*,*)'***************************************************
WRITE(*,*) 'Computing the three_stress stochastic constraints' 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1500) PHI_P 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1510) G_SIGMA_CONSTRAINT(1) 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1520) G_SIGMA_CONSTRAINT(2) 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1530) G_SIGMA_CONSTRAINT(3) 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,*) 'Computing two_displacement stochastic constraint' 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1540) G_X_CONSTRAINT(1) 
WRITE(*,*) '-----------------------------------------------' 
WRITE(*,*) '  ' 
WRITE(*,1550) G_X_CONSTRAINT(2) 
WRITE(*,*) '  ' 
WRITE(*,*) '***************************************************' 
WRITE(*,*) '***************************************************' 
WRITE(*,*) '  ' 
WRITE(*,*) 'Computing the mean value of weight of structure' 
WRITE(*,*) '  ' 
WRITE(*,1560) WEIGHT 
WRITE(*,*) '-----------------------------------------------' 

800   FORMAT(1X,'N=',I3,4X,'M=',I3//) 
810   FORMAT(3(2X,D20.10)/) 
820   FORMAT(3(2X,D20.10)/) 
830   FORMAT(3(2X,D20.10)/) 
840   FORMAT(3(2X,D20.10)/) 
850   FORMAT(3(2X,D20.10)/) 
860   FORMAT(2X,D20.10/) 
870   FORMAT(2X,D20.10/) 
880   FORMAT(2X,D20.10/) 
890   FORMAT(2X,D20.10/) 
900   FORMAT(2X,D20.10/) 
910   FORMAT(10(1X,D10.5)/) 
920   FORMAT(3(2X,D20.10)/) 
930   FORMAT(3(2X,D20.10)/) 
940   FORMAT(3(2X,D20.10)/) 
950   FORMAT(3(2X,D20.10)/) 
960   FORMAT(3(2X,D20.10)/) 
970   FORMAT(3(2X,D20.10)/) 
980   FORMAT(3(2X,D20.10)/) 
990   FORMAT(3(2X,D20.10)/)
C     Calculating the inverse matrix A
SUBROUTINE MAT_INV(A,N,L,IS,JS)
DIMENSION A(N,N), IS(N), JS(N)
DOUBLE PRECISION A, T, D
L = 1
DO 90 K = 1,N
   D = 0.0
   DO 10 I = K, N
      DO 10 J = K, N
         IF (ABS(A(I,J)).GT.D) THEN
            D = ABS(A(I,J))
            IS(K) = I
            JS(K) = J
         ENDIF
10      CONTINUE
   IF (D+1.0.EQ.1.0) THEN
      L = 0
   ENDIF
90      CONTINUE
END
WRITE(*,20)
RETURN
END IF
20 FORMAT(1X,'ERR *** NOT INV')
DO 30 J = 1,N
   T = A(K,J)
   A(K,J) = A(IS(K),J)
   A(IS(K),J) = T
30 CONTINUE
DO 40 I = 1,N
   T = A(I,K)
   A(I,K) = A(I,JS(K))
   A(I,JS(K)) = T
40 CONTINUE
A(K,K) = 1/A(K,K)
DO 50 J = 1,N
   IF (J.NE.K) THEN
      A(K,J) = A(K,J)*A(K,K)
   END IF
50 CONTINUE
DO 70 I = 1,N
   IF (I.NE.K) THEN
      DO 60 J = 1,N
         IF (J.NE.K) THEN
            A(I,J) = A(I,J) - A(I,K)*A(K,J)
         END IF
60      CONTINUE
   END IF
70 CONTINUE
DO 80 I = 1,N
   IF (I.NE.K) THEN
      A(I,K) = -A(I,K)*A(K,K)
   END IF
80 CONTINUE
90 CONTINUE
DO 120 K = N, 1, -1
   DO 100 J = 1,N
      T = A(K,J)
      A(K,J) = A(JS(K),J)
      A(JS(K),J) = T
100 CONTINUE
DO 110 I = 1,N
   T = A(I,K)
   A(I,K) = A(I,IS(K))
   A(I,IS(K)) = T
110 CONTINUE
120 CONTINUE
RETURN
END

C      Buliding up the first,second derivative of the inital deformation vector [Beta,i] and [Beta(t),ij]
SUBROUTINE BETA_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA, *
                           AREA,E,P,BETA_F)
   DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
233
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), BETA_F(N,1)
IF (I.EQ.5.OR.I.EQ.8) THEN
  BETA_F(1,1) = ALPHA*TEMP*LENGTH(1)/2.0
  BETA_F(2,1) = ALPHA*TEMP*LENGTH(2)/2.0
  BETA_F(3,1) = ALPHA*TEMP*LENGTH(3)/2.0
ELSE IF (I.EQ.9) THEN
  BETA_F(1,1) = DELTA(1)/SQRT(2.0)
  BETA_F(2,1) = 0.D0
  BETA_F(3,1) = 0.D0
ELSE IF (I.EQ.10) THEN
  BETA_F(1,1) = -DELTA(2)/SQRT(2.0)
  BETA_F(2,1) = 0.D0
  BETA_F(3,1) = 0.D0
ELSE
  BETA_F(1,1) = 0.D0
  BETA_F(2,1) = 0.D0
  BETA_F(3,1) = 0.D0
ENDIF
RETURN
END

SUBROUTINE BETA_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
  AREA,E,P,BETA_S)
DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), BETA_S(N,1)
IF ((I.EQ.5.AND.J.EQ.8).OR.(I.EQ.8.AND.J.EQ.5)) THEN
  BETA_S(1,1) = ALPHA*TEMP*LENGTH(1)/2.0
  BETA_S(2,1) = ALPHA*TEMP*LENGTH(2)/2.0
  BETA_S(3,1) = ALPHA*TEMP*LENGTH(3)/2.0
ELSE
  BETA_S(1,1) = 0.D0
  BETA_S(2,1) = 0.D0
  BETA_S(3,1) = 0.D0
ENDIF
RETURN
END

SUBROUTINE P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
  AREA,E,P,P_F)
DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), P_F(N,1)
IF ((I.EQ.5).OR.(I.EQ.8)) THEN
  P_F(1,1) = 0.D0
  P_F(2,1) = 0.D0
  P_F(3,1) = SQRT(2.0)*ALPHA*TEMP*(-LENGTH(1)+SQRT(2.0)*LENGTH(2)
    -LENGTH(3))/4.0
ELSE IF (I.EQ.6) THEN
  P_F(1,1) = P(1)
  P_F(2,1) = 0.D0
  P_F(3,1) = 0.D0
ELSE IF (I.EQ.7) THEN
  P_F(1,1) = 0.D0
  P_F(2,1) = P(2)
  P_F(3,1) = 0.D0
ENDIF
RETURN
END

C Building up the first, second derivative of the mechanical load vector [P,i] and [P,ij]
SUBROUTINE P_FIRST_DER(I,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,AREA,E,
  P,P_F)
DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), P_F(N,1)
IF ((I.EQ.5).OR.(I.EQ.8)) THEN
  P_F(1,1) = 0.D0
  P_F(2,1) = 0.D0
  P_F(3,1) = SQRT(2.0)*ALPHA*TEMP*(-LENGTH(1)+SQRT(2.0)*LENGTH(2)
    -LENGTH(3))/4.0
ELSE IF (I.EQ.6) THEN
  P_F(1,1) = P(1)
  P_F(2,1) = 0.D0
  P_F(3,1) = 0.D0
ELSE IF (I.EQ.7) THEN
  P_F(1,1) = 0.D0
  P_F(2,1) = P(2)
  P_F(3,1) = 0.D0
ENDIF
RETURN
END
ELSE IF (I.EQ.9) THEN
    P_F(1,1) = 0.D0
    P_F(2,1) = 0.D0
    P_F(3,1) = -DELTA(1)/2.0
ELSE IF (I.EQ.10) THEN
    P_F(1,1) = 0.D0
    P_F(2,1) = 0.D0
    P_F(3,1) = DELTA(2)/2.0
ELSE
    P_F(1,1) = 0.D0
    P_F(2,1) = 0.D0
    P_F(3,1) = 0.D0
ENDIF
RETURN
END

SUBROUTINE P_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
*                                                        AREA,E,P,P_S)
DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), P_S(N,1)
IF ((I.EQ.5.AND.J.EQ.8).OR.(I.EQ.8.AND.J.EQ.5)) THEN
    P_S(1,1) = 0.D0
    P_S(2,1) = 0.D0
    P_S(3,1) = SQRT(2.0)*ALPHA*TEMP*(-LENGTH(1)+SQRT(2.0)*LENGTH(2)
*                         -LENGTH(3))/4.0
ELSE
    P_S(1,1) = 0.D0
    P_S(2,1) = 0.D0
    P_S(3,1) = 0.D0
ENDIF
RETURN
END

SUBROUTINE P_SECOND_DER(I,J,N,LENGTH,DEPTH,ALPHA,TEMP,DELTA,
*                                                        AREA,E,P,P_S)
DOUBLE PRECISION LENGTH(N), DEPTH(2), ALPHA, TEMP
DOUBLE PRECISION DELTA(2), AREA(N), E, P(2), P_S(N,1)
IF ((I.EQ.5.AND.J.EQ.8).OR.(I.EQ.8.AND.J.EQ.5)) THEN
    P_S(1,1) = 0.D0
    P_S(2,1) = 0.D0
    P_S(3,1) = SQRT(2.0)*ALPHA*TEMP*(-LENGTH(1)+SQRT(2.0)*LENGTH(2)
*                         -LENGTH(3))/4.0
ELSE
    P_S(1,1) = 0.D0
    P_S(2,1) = 0.D0
    P_S(3,1) = 0.D0
ENDIF
RETURN
END

C Buliding up the first, second derivative of area matrix [A,i] and [A,ij]
SUBROUTINE AREA_FIRST_DER(I,N,AREA_BAR,AREA_F)
DOUBLE PRECISION AREA_BAR(N,N), AREA_F(N,N)
DO 10 J = 1,N
    DO 10 K = 1,N
        AREA_F(J,K) = 0.D0
10  CONTINUE
IF (I.EQ.1) THEN
    AREA_F(1,1) = AREA_BAR(1,1)
ELSE IF (I.EQ.2) THEN
    AREA_F(2,2) = AREA_BAR(2,2)
ELSE IF (I.EQ.3) THEN
    AREA_F(3,3) = AREA_BAR(3,3)
ENDIF
RETURN
END

SUBROUTINE AREA_SECOND_DER(I,J,N,AREA_BAR,AREA_S)
DOUBLE PRECISION AREA_BAR(N,N), AREA_S(N,N)
DO 10 K = 1,N
    DO 10 L = 1,N
        AREA_S(K,L) = 0.D0
10  CONTINUE
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C Building up the first, second derivative of the Young's modulus matrix \([E,i]\) and \([E,ij]\)

SUBROUTINE E_FIRST_DER(I,N,E_BAR,E_F)
DOUBLE PRECISION E_BAR(N,N), E_F(N,N)
DO 10 J = 1,N
   DO 10 K = 1,N
      E_F(J,K) = 0.D0
   10 CONTINUE
   IF (I.EQ.4) THEN
      E_F(1,1) = E_BAR(1,1)
      E_F(2,2) = E_BAR(2,2)
      E_F(3,3) = E_BAR(3,3)
   ENDIF
   RETURN
END

SUBROUTINE E_SECOND_DER(I,J,N,E_BAR,E_S)
DOUBLE PRECISION E_BAR(N,N), E_S(N,N)
DO 10 K = 1,N
   DO 10 L = 1,N
      E_S(K,L) = 0.D0
   10 CONTINUE
   RETURN
END

C Building up the first, second derivative of flexibility matrix \([G,i]\) and \([G,ij]\)

SUBROUTINE G_FIRST_DER(I,N,G_BAR,G_F)
DOUBLE PRECISION G_BAR(N,N), G_F(N,N)
DO 10 J = 1,N
   DO 10 K = 1,N
      G_F(J,K) = 0.D0
   10 CONTINUE
   IF (I.EQ.1) THEN
      G_F(1,1) = -G_BAR(1,1)
   ELSE IF (I.EQ.2) THEN
      G_F(2,2) = -G_BAR(2,2)
   ELSE IF (I.EQ.3) THEN
      G_F(3,3) = -G_BAR(3,3)
   ELSE IF (I.EQ.4) THEN
      G_F(1,1) = -G_BAR(1,1)
      G_F(2,2) = -G_BAR(2,2)
      G_F(3,3) = -G_BAR(3,3)
   ENDIF
   RETURN
END

SUBROUTINE G_SECOND_DER(I,J,N,G_BAR,G_S)
DOUBLE PRECISION G_BAR(N,N), G_S(N,N)
DO 10 K = 1,N
   DO 10 L = 1,N
      G_S(K,L) = 0.D0
   10 CONTINUE
IF (I.EQ.1.AND.J.EQ.1) THEN
   G_S(1,1) = 2*G_BAR(1,1)
ELSE IF ((I.EQ.1.AND.J.EQ.4).OR.(I.EQ.4.AND.J.EQ.1)) THEN
   G_S(2,2) = 2*G_BAR(2,2)
ELSE IF ((I.EQ.2.AND.J.EQ.4).OR.(I.EQ.4.AND.J.EQ.2)) THEN
   G_S(3,3) = 2*G_BAR(3,3)
ELSE IF (I.EQ.3.AND.J.EQ.3) THEN
   G_S(3,3) = 2*G_BAR(3,3)
ELSE IF (I.EQ.4.AND.J.EQ.4) THEN
   G_S(1,1) = 2*G_BAR(1,1)
   G_S(2,2) = 2*G_BAR(2,2)
   G_S(3,3) = 2*G_BAR(3,3)
ENDIF
RETURN
END

C Building up the first, second derivative of governing matrix [S,i] and [S,ij]
SUBROUTINE S_FIRST_DER(I,N,S_BAR,S_F)
DOUBLE PRECISION S_BAR(N,N), S_F(N,N)
DO 10 J = 1,N
   DO 10 K = 1,N
      S_F(J,K) = 0.D0
   10 CONTINUE
IF (I.EQ.1) THEN
   S_F(3,1) = -S_BAR(3,1)
ELSE IF (I.EQ.2) THEN
   S_F(3,2) = -S_BAR(3,2)
ELSE IF (I.EQ.3) THEN
   S_F(3,3) = -S_BAR(3,3)
ELSE IF (I.EQ.4) THEN
   S_F(3,1) = -S_BAR(3,1)
   S_F(3,2) = -S_BAR(3,2)
   S_F(3,3) = -S_BAR(3,3)
ENDIF
RETURN
END

SUBROUTINE S_SECOND_DER(I,J,N,S_BAR,S_S)
DOUBLE PRECISION S_BAR(N,N), S_S(N,N)
DO 10 K = 1,N
   DO 10 L = 1,N
      S_S(K,L) = 0.D0
   10 CONTINUE
IF (I.EQ.1.AND.J.EQ.1) THEN
   S_S(3,1) = 2*S_BAR(3,1)
ELSE IF ((I.EQ.1.AND.J.EQ.4).OR.(I.EQ.4.AND.J.EQ.1)) THEN
   S_S(2,2) = 2*S_BAR(2,2)
ELSE IF ((I.EQ.2.AND.J.EQ.4).OR.(I.EQ.4.AND.J.EQ.2)) THEN
   S_S(3,3) = 2*S_BAR(3,3)
ELSE IF (I.EQ.3.AND.J.EQ.3) THEN
   S_S(3,3) = 2*S_BAR(3,3)
ELSE IF (I.EQ.4.AND.J.EQ.4) THEN
   S_S(1,1) = 2*S_BAR(1,1)
   S_S(2,2) = 2*S_BAR(2,2)
   S_S(3,3) = 2*S_BAR(3,3)
ENDIF
RETURN
END
ELSE IF (I.EQ.3.AND.J.EQ.3) THEN
   S_S(3,3) = 2*S_BAR(3,3)
   S_S(3,3) = S_BAR(3,3)
ELSE IF (I.EQ.4.AND.J.EQ.4) THEN
   S_S(3,1) = 2*S_BAR(3,1)
   S_S(3,2) = 2*S_BAR(3,2)
   S_S(3,3) = 2*S_BAR(3,3)
ENDIF
RETURN
END

3. Maple code of the stochastic analysis for example 10

> restart;
> settime := time():
> n := 3: m := 2: NN := 10:
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> rho :=
   matrix(10,10,
   [40/4000,20/4000,5/4000,0,0,0,0,0,0,0,20/4000,40/4000,5/4000,0,0,0,0,0,0,0,5/4000,5/4000,1
   0,40000,0,0,0,0,0,0,40/4000,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
> for j from 1 to NN do for i from 1 to NN do  \( \beta_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( \beta_{\text{second} \_\text{der}}[0][i,j] \))) od od:

> 2) Builting up the Deterministic value, the First,second derivatives of the Total Load: \( P_{\text{bar}} \), \( P_{\text{first} \_\text{der}} \)
> and \( P_{\text{second} \_\text{der}} \).
> \( P_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(P[\text{total}])) \):
> for i from 1 to NN do  \( P_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, P[\text{total}], q[i]))) \) od:
> for j from 1 to NN do \( P_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(\text{map}(\text{diff}, P[\text{total}], q[i], q[j]), i=1..NN), \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( P_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( P_{\text{second} \_\text{der}}[i,j] \))) od od:

> 3) Builting up the Deterministic value, the First,second derivatives of the \( Y \) matrix: \( Y_{\text{bar}} \), \( Y_{\text{bar} \_\text{inv}} \),
> \( Y_{\text{first} \_\text{der}} \) and \( Y_{\text{second} \_\text{der}} \).
> \( Y_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(Y)) \):
> \( Y_{\text{bar} \_\text{inv}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{inverse}(Y)) \):
> for i from 1 to NN do  \( Y_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, Y, q[i]))) \) od:
> for j from 1 to NN do \( Y_{\text{second}}[j] := \text{seq}(\text{map}(\text{diff}, Y, q[i], q[j]), i=1..NN) \) od:
> \( Y_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(Y_{\text{second}}[i]), i=1..NN], \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( Y_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( Y_{\text{second} \_\text{der}}[i,j] \))) od od:

> 4) Builting up the Deterministic value, the First,second derivatives of the \( Z \) matrix: \( Z_{\text{bar}} \), \( Z_{\text{bar} \_\text{inv}} \),
> \( Z_{\text{first} \_\text{der}} \) and \( Z_{\text{second} \_\text{der}} \).
> \( Z_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(Z)) \):
> \( Z_{\text{bar} \_\text{inv}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{inverse}(Z)) \):
> for i from 1 to NN do  \( Z_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, Z, q[i]))) \) od:
> for j from 1 to NN do \( Z_{\text{second}}[j] := \text{seq}(\text{map}(\text{diff}, Z, q[i], q[j]), i=1..NN) \) od:
> \( Z_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(Z_{\text{second}}[i]), i=1..NN], \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( Z_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( Z_{\text{second} \_\text{der}}[i,j] \))) od od:

> 5) Builting up the Deterministic value, the First,second derivatives of the Flexibility matrix:
> \( G_{\text{bar}}, G_{\text{bar} \_\text{inv}}, G_{\text{first} \_\text{der}} \) and \( G_{\text{second} \_\text{der}} \).
> \( G_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(G)) \):
> \( G_{\text{bar} \_\text{inv}} := \text{inverse}(G_{\text{bar}}) \):
> for i from 1 to NN do  \( G_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, G, q[i]))) \) od:
> for j from 1 to NN do \( G_{\text{second}}[j] := \text{seq}(\text{map}(\text{diff}, G, q[i], q[j]), i=1..NN) \) od:
> \( G_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(G_{\text{second}}[i]), i=1..NN], \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( G_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( G_{\text{second} \_\text{der}}[i,j] \))) od od:

> 6) Builting up the Deterministic value, the First,second derivatives of the Governing matrix:
> \( S_{\text{bar}}, S_{\text{bar} \_\text{inv}}, S_{\text{first} \_\text{der}} \) and \( S_{\text{second} \_\text{der}} \).
> \( S_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(S)) \):
> \( S_{\text{bar} \_\text{inv}} := \text{inverse}(S_{\text{bar}}) \):
> for i from 1 to NN do  \( S_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, S, q[i]))) \) od:
> for j from 1 to NN do \( S_{\text{second}}[j] := \text{seq}(\text{map}(\text{diff}, S, q[i], q[j]), i=1..NN) \) od:
> \( S_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(S_{\text{second}}[i]), i=1..NN], \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( S_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( S_{\text{second} \_\text{der}}[i,j] \))) od od:

> Builting up the Deformation Coefficient Matrix \( [J_{\text{bar}}] \)
> \( J_{\text{bar}} := \text{submatrix}(%) \)

> Construction for the Derivatives of the Random Variables in IFMD
> 1) Builting up the Deterministic value, the First,second derivatives of the Dual matrix: \( D_{\text{bar}}, D_{\text{bar} \_\text{inv}}, D_{\text{first} \_\text{der}} \) and \( D_{\text{second} \_\text{der}} \).
> \( D := \text{multiply}(\text{inverse}(G), \text{transpose}(B)) \):
> \( D_{\text{bar}} := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(D)) \):
> \( D_{\text{bar} \_\text{inv}} := \text{inverse}(D_{\text{bar}}) \):
> for i from 1 to NN do  \( D_{\text{first} \_\text{der}}[i] := \text{subs}({\text{seq}(q[i]=0,i=1..NN)},\text{evalm}(\text{map}(\text{diff}, D, q[i]))) \) od:
> for j from 1 to NN do \( D_{\text{second}}[j] := \text{seq}(\text{map}(\text{diff}, D, q[i], q[j]), i=1..NN) \) od:
> \( D_{\text{second} \_\text{der}} := \text{convert}([\text{seq}(D_{\text{second}}[i]), i=1..NN], \text{matrix}] \) od:
> for j from 1 to NN do for i from 1 to NN do  \( D_{\text{second} \_\text{der}}[i,j] \)
> :=subs({seq(q[k]=0,k=1..NN)},evalm(\( D_{\text{second} \_\text{der}}[i,j] \))) od od:
for j from 1 to NN do Dr_second[j] := seq(map(diff,Dr,q[i],q[j]),i=1..NN) od:
> Dr_second_der := convert([ seq(Dr_second[i],i=1..NN)],matrix):
> for j from 1 to NN do for i from 1 to NN do Dr_second_der[i,j] := subs(seq(q[k]=0,k=1..NN),evalm(Dr_second_der[i,j])) od od:

2) Building up the Deterministic value, the First, second derivatives of the Governing matrix: G_bar, G_first_der and G_second_der.

> P_D := evalm(P[mech]+multiply(B,inverse(G),beta[total])):
> P_D_bar := subs(seq(q[i]=0,i=1..NN),evalm(P_D)):
> for i from 1 to NN do P_D_first_der[i] := subs(seq(q[i]=0,i=1..NN),evalm(map(diff, P_D, q[i]))) od:
> for j from 1 to NN do P_D_second[i,j] := seq(map(diff, P_D, q[i], q[j]), i=1..NN) od:
> P_D_second_der := convert([ seq(P_D_second[i,j], i=1..NN)], matrix):
> for j from 1 to NN do for i from 1 to NN do P_D_second_der[i,j] := subs(seq(q[k]=0,k=1..NN),evalm(P_D_second_der[i,j])) od od:

Calculation in IFM

1) Calculating the perturbation equations of Force vector (F_bar, F_first_der[i], F_second_der[i,j]) in IFM

> F_bar := multiply(S_bar_inv,P_bar):
> for i from 1 to NN do F_first_der[i] := subs(seq(q[i]=0,i=1..NN),multiply(S_bar_inv,evalm(P_first_der[i]-multiply(S_first_der[i],F_bar)))) od:
> for j from 1 to NN do for i from 1 to NN do F_second_der[i,j] := subs(seq(q[k]=0,k=1..NN),multiply(S_bar_inv,evalm(P_second_der[i,j]-2*multiply(S_first_der[i],F_first_der[j])-multiply(S_second_der[i,j],F_bar)))) od od:

the first-order approximation:

mu[F_1st] := subs(seq(q[i]=0,i=1..NN),evalm(F_bar)):
> cov[F_1st] := sum(sum(rho[i,j]*multiply(F_first_der[i],transpose(F_first_der[j])),i=1..NN),j=1..NN):
> the second-order approximation:

mu[F_2nd] := evalm(F_bar+0.5*sum(sum(rho[i,j]*F_second_der[i,j]),i=1..NN),j=1..NN)):
> cov[F_2nd] := sum(sum(rho[i,j]*multiply(F_first_der[i],transpose(F_first_der[j])),i=1..NN),j=1..NN)):

2) Calculating the perturbation equations of Displacement vector (X_bar, X_first_der[i], X_second_der[i,j]) in IFM

> X_bar := multiply(J_bar,evalm(multiply(G_bar,S_bar_inv,P_bar)+beta_bar_0)):
> for i from 1 to NN do X_first_der[i] := subs(seq(q[i]=0,i=1..NN),multiply(J_bar,evalm(multiply(G_bar,F_first_der[i])+multiply(G_first_der[i],F_bar)+beta_first_der_0[i]))) od:
> for j from 1 to NN do for i from 1 to NN do X_second_der[i,j] := subs(seq(q[k]=0,k=1..NN),multiply(J_bar,evalm(multiply(G_bar,F_second_der[i,j])+2*multiply(G_first_der[i],F_first_der[j])+multiply(G_second_der[i,j],F_bar)+beta_second_der_0[i,j]))) od od:

the first-order approximation:

mu[X_1st] := subs(seq(q[i]=0,i=1..NN),evalm(X_bar)):
> cov[X_1st] := sum(sum(rho[i,j]*multiply(X_first_der[i],transpose(X_first_der[j])),i=1..NN),j=1..NN):
> the second-order approximation:

mu[X_2nd] := evalm(X_bar+0.5*sum(sum(rho[i,j]*X_second_der[i,j]),i=1..NN),j=1..NN)):
> cov[X_2nd] := sum(sum(rho[i,j]*multiply(X_first_der[i],transpose(X_first_der[j])),i=1..NN),j=1..NN):

Calculation in IFMD

1) Calculating the perturbation equations of Displacement vector (X_D_bar, X_D_first_der[i], X_D_second_der[i,j]) in IFMD

> X_D_bar := multiply(Dr_bar_inv,P_D_bar):
> for i from 1 to NN do X_D_first_der[i] := subs(seq(q[i]=0,i=1..NN),multiply(Dr_bar_inv,evalm(P_D_first_der[i]-multiply(Dr_first_der[i],X_D_bar)))) od:
> for j from 1 to NN do for i from 1 to NN do X_D_second_der[i,j] := subs(seq(q[k]=0,k=1..NN),evalm(P_D_second_der[i,j]-2*multiply(Dr_first_der[i],X_D_first_der[j])-multiply(Dr_second_der[i,j],X_D_bar))) od od:
for j from 1 to NN do for i from 1 to NN do  X_D_second_der[i,j] := subs({seq(q[k]=0,k=1..NN)},multiply(Dr_bar_inv,evalm(P_D_second_der[i,j]-
2*multiply(Dr_first_der[i],X_D_first_der[j])-multiply(Dr_second_der[i,j],X_D_bar)))) od od:
> the first-order approximation:
> mu[X_D_1st] := subs({seq(q[i]=0,i=1..NN)},evalm(X_D_bar)):
> cov[X_D_1st] :=
sum('sum('rho[i,j]*multiply(X_D_first_der[i],transpose(X_D_first_der[j]))','i'=1..NN)','j'=1..NN):
> the second-order approximation:
> mu[X_D_2nd] := evalm(X_D_bar+0.5*sum('sum('rho[i,j]*X_D_second_der[i,j]','i'=1..NN)','j'=1..NN)):
> cov[X_D_2nd] :=
sum('sum('rho[i,j]*multiply(X_D_first_der[i],transpose(X_D_first_der[j]))','i'=1..NN)','j'=1..NN):
> 2) Calculating the perturbation equations of Force vector (F_D_bar, F_D_first_der[i],F_D_second_der[i,j])in IFMD
> F_D_bar := multiply(G_bar_inv,evalm(multiply(transpose(B),X_D_bar)-beta_bar_0)):
> for i from 1 to NN do  F_D_first_der[i] :=
subs({seq(q[i]=0,i=1..NN)},multiply(G_bar_inv,evalm(multiply(transpose(B),X_D_first_der[i])-
beta_first_der_0[i]-multiply(G_first_der[i],F_D_bar)))) od:
> for j from 1 to NN do for i from 1 to NN do  F_D_second_der[i,j] :=
subs({seq(q[k]=0,k=1..NN)},multiply(G_bar_inv,evalm(multiply(transpose(B),X_D_second_der[i,j])-
2*multiply(G_first_der[i],F_D_first_der[j])-
multiply(G_second_der[i,j],F_D_bar)))) od od:
> the first-order approximation:
> mu[F_D_1st] := subs({seq(q[i]=0,i=1..NN)},evalm(F_D_bar)):
> cov[F_D_1st] :=
sum('sum('rho[i,j]*multiply(F_D_first_der[i],transpose(F_D_first_der[j]))','i'=1..NN)','j'=1..NN):
> the second-order approximation:
> mu[F_D_2nd] := evalm(F_D_bar+0.5*sum('sum('rho[i,j]*F_D_second_der[i,j]','i'=1..NN)','j'=1..NN)):
> cov[F_D_2nd] :=
sum('sum('rho[i,j]*multiply(F_D_first_der[i],transpose(F_D_first_der[j]))','i'=1..NN)','j'=1..NN):
> 3) Calculating the perturbation equations of stress vector (sigma_bar, sigma_first_der[i],sigma_second_der[i,j])in IFM
> sigma_bar := multiply(Y_bar,F_bar):
> for i from 1 to NN do  sigma_first_der[i] :=
subs({seq(q[i]=0,i=1..NN)},evalm(multiply(Y_bar,F_first_der[i])+multiply(Y_first_der[i],F_bar)))  od:
> for j from 1 to NN do for i from 1 to NN do  sigma_second_der[i,j] :=
subs({seq(q[k]=0,k=1..NN)},evalm(multiply(Y_bar,F_second_der[i,j])+2*multiply(Y_first_der[i],F_first_der[j])+
multiply(Y_second_der[i,j],sigma_first_der[i,j]))
4) Calculating the perturbation equations of strain vector (epsilon_bar, epsilon_first_der[i],epsilon_second_der[i,j])in IFM
> epsilon_bar := multiply(Z_bar,sigma_bar):
> for i from 1 to NN do  epsilon_first_der[i] :=
subs({seq(q[i]=0,i=1..NN)},evalm(multiply(Z_bar,sigma_first_der[i])+multiply(Z_first_der[i],sigma_bar)))  od:
> for j from 1 to NN do for i from 1 to NN do  epsilon_second_der[i,j] :=
subs({seq(q[k]=0,k=1..NN)},evalm(multiply(Z_bar,sigma_second_der[i,j])+2*multiply(Z_first_der[i],sigma_first_der[i,j])
+multiply(Z_second_der[i,j],sigma_bar))) od od:
> the first-order approximation
> \[
> \mu_{\epsilon_{1st}} := \text{subs}\left(\{\text{seq}(q[i]=0, i=1..\text{NN})\}, \text{evalm}(\epsilon_{bar})\right);
> \]
> \[
> \text{cov}_{\epsilon_{1st}} := \text{sum}\left(\text{sum}\left(\rho[i,j] \cdot \text{multiply}(\epsilon_{first\_der}[i], \text{transpose}(\epsilon_{first\_der}[j])), i=1..\text{NN}\right), j=1..\text{NN}\right);
> \]
> the second-order approximation:
> \[
> \mu_{\epsilon_{2nd}} := \text{evalm}(\epsilon_{bar} + 0.5 \cdot \text{sum}\left(\text{sum}\left(\rho[i,j] \cdot \epsilon_{second\_der}[i,j], i=1..\text{NN}\right), j=1..\text{NN}\right));
> \]
> \[
> \text{cov}_{\epsilon_{2nd}} := \text{sum}\left(\text{sum}\left(\rho[i,j] \cdot \text{multiply}(\epsilon_{first\_der}[i], \text{transpose}(\epsilon_{first\_der}[j])), i=1..\text{NN}\right), j=1..\text{NN}\right);
> \]

> IFM and IFMD Computation for The Three-bar truss
> 1) the deterministic, mean values and ratio of Force vector
> \[
> \]
> \[
> F_{\text{bar\_value}} := [62.7803471310, -7.9303309877]
> \]

> \]
> \[
> \text{mu}[F_{1st\_value}] := [62.7803471310, -7.9303309877]
> \]

> \]
> \[
> \mu[F_{D_1st\_value}] := [62.7803471309, -7.93033098783]
> \]

> \]
> \[
> \text{cov}[F_{1st\_value}] := 
> \begin{bmatrix}
> 19.2671363130 & 1.78128539729 & -4.47655039465 \\
> 1.78128539729 & 18.9274914395 & -8.83424466421 \\
> -4.47655039465 & -8.83424466421 & 21.7797628977
> \end{bmatrix}
> \]

> \]
> \[
> \text{cov}[F_{D_1st\_value}] := 
> \begin{bmatrix}
> 19.2671363130 & 1.78128539735 & -4.47655039469 \\
> 1.78128539735 & 18.9274914394 & -8.83424466412 \\
> -4.47655039469 & -8.83424466412 & 21.7797628977
> \end{bmatrix}
> \]

> \]
> \[
> \text{mu}[F_{2nd\_value}] := [62.7647183246, -7.94595979413]
> \]

> \]
> \[
> \text{cov}[F_{2nd\_value}] := 
> \begin{bmatrix}
> 19.2671363130 & 1.78128539729 & -4.47655039465 \\
> 1.78128539729 & 18.9274914395 & -8.83424466421 \\
> -4.47655039465 & -8.83424466421 & 21.7797628977
> \end{bmatrix}
> \]
\[
\begin{bmatrix}
1.78128539729 & 18.9274914395 & -8.8342466421 \\
-4.47655039465 & -8.8342466421 & 21.7797628977
\end{bmatrix}
\]

\[
\text{cov}[F_\text{D,2nd_value}] := \\
\begin{bmatrix}
19.2671363130 & 1.78128539735 & -4.47655039469 \\
1.78128539735 & 18.9274914394 & -8.83424466412 \\
-4.47655039469 & -8.83424466412 & 21.7797628977
\end{bmatrix}
\]

\[> \text{the ratio of force vector in the second-order approximation} \]

\[
\text{delta}[\mu[F]] := \text{evalm}(\mu[F_\text{2nd_value}]-F_\text{bar_value})
\]

\[
\text{for i from 1 to n do if (mu[F_2nd_value][i,1]=0) then R[F][i,1] := 0 else R[F][i,1] := 100*delta[mu[F]][i,1]/mu[F_2nd_value][i,1] fi od;}
\]

\[
\text{print(R[F]);}
\]

\[
\text{table}\left(\left(3, 1\right) = 0.1966887123, \left(2, 1\right) = 0.03609315848, \left(1, 1\right) = -0.02490062955 \right)
\]

\[> \text{the correlation coefficients and percentage of covariance of force in the first- and second-order approximation} \]

\[
\text{Std}_F := [\text{seq(cov[F_2nd_value][i,i], i=1..n)}] ;
\]

\[
\text{Std\_dev}_F := \text{map(sqrt,Std\_dev}_F) ;
\]

\[
\text{for j from 1 to n do for i from 1 to n do if (Std\_dev}_F[i]=0 or Std\_dev}_F[j]=0  ) then Rho[i,j] := 0 else Rho[i,j] := cov[F_2nd_value][i,j]/(Std\_dev}_F[i]*Std\_dev}_F[j]) fi od od;}
\]

\[
\text{matrix(n,n,(i,j)->Rho[i,j]);}
\]

\[
\begin{bmatrix}
1.000000000 & 0.09327782123 & -0.2185286179 \\
0.09327782123 & 1.000000000 & -0.4351072501 \\
-0.2185286179 & -0.4351072501 & 1.000000000
\end{bmatrix}
\]

\[> \text{C_temp := [seq(mu[F_2nd_value][i,1], i=1..n)}] ;
\]

\[
\text{B_temp := dotprod(C_temp, C_temp)} ;
\]

\[
\text{cov}_F\text{\_abs} := \text{map(abs,cov[F_2nd_value])} ;
\]

\[
\text{cov}_F\text{\_percent} := \text{evalm(100*map(sqrt, evalm(cov}_F\text{\_abs/B_temp))}} ;
\]

\[
\text{cov}_F\text{\_percent} := [4.985241594, 1.515808642, 2.402975616] ;
\]

\[> 2) \text{the deterministic and the first-order approximation of Displacement vector} \]

\[
\]

\[
\text{X_bar_value} := [-0.237050605457, 0.197485042083]
\]

\[
\]

\[
\text{mu}[X_\text{1st_value}] := [-0.237050605457, 0.197485042083]
\]

\[
\]

\[
\text{mu}[X_\text{D,1st_value}] := [-0.237050605457, 0.197485042083]
\]

\[
\]

\[
\text{cov}[X_\text{1st_value}] := [-0.00893832812963, 0.000869953094870 , -0.000893832812963] ;
\]

\[
\]

\[
\text{cov}[X_\text{D,1st_value}] := [-0.00893832812965, 0.000869953094865 , -0.000893832812965] ;
\]

\[
\]

\[
\text{mu}[X_\text{2nd_value}] := [0.201456412369]
\]

\[243\]
mu[X_2nd_value] := [-.240676001670]
> the ratio of displacement vector in the second-order approximation
> delta[mu[X]] := evalm(mu[X_2nd_value]-X_bar_value):
for i from 1 to m do if (mu[X_2nd_value][i,1]=0) then R[X][i,1] := 0 else R[X][i,1] := 100*delta[mu[X]][i,1]/mu[X_2nd_value][i,1] fi od;
> print(R[X]);
table(
(2, 1) = 1.506338885,
(1, 1) = 1.971329804
)
> the correlation coefficients and percentage of covariance of displacement in the first- and second-order approximation
> Std_X := [seq(cov[X_2nd_value][i,i], i=1..m )]:
> Std_dev_X := map(sqrt,Std_X):
> for j from 1 to m do for i from 1 to m do  if (Std_dev_X[i]=0 or Std_dev_X[j]=0  ) then Rho[i,j] := 0 else Rho[i,j] := cov[X_2nd_value][i,j]/(Std_dev_X[i]*Std_dev_X[j]) fi od od;
> matrix(m,m,(i,j)->Rho[i,j]);
[.9999999997 , -.8662919459]
[.8662919457 , 1.000000000]
> C_temp := [seq(mu[X_2nd_value][i,1], i=1..m )]:
> B_temp := dotprod(C_temp, C_temp):
> cov_X_abs := map(abs,cov[X_2nd_value]):
> cov_X_percent := evalm(100*map(sqrt, evalm(cov_X_abs/B_temp)));
> 3) the deterministic and the first-order approximation of Stress vector
> sigma_bar_value := [62.7803471310, -3.9651654939]
> mu[sigma_1st_value] := [62.7803471310, -3.9651654939]
> cov[sigma_1st_value] := [45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831, 45.0325770831]
\[
\text{cov}[\sigma_{2nd\text{\_value}}] := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf}('\text{cov}[\sigma_{2nd\text{\_value}}]), 12); \\
\text{cov}[\sigma_{2nd\text{\_value}}] :=
\begin{bmatrix}
45.0325770831 & 37.0167023515 & -5.97233884362 \\
37.0167023515 & 38.3860130083 & -1.52068551119 \\
-5.97233884362 & -1.52068551119 & 5.47342127648
\end{bmatrix}
\]

\(> \) the ratio of stress vector in the second-order approximation

\(> \) for i from 1 to n do if (\text{mu}[\sigma_{2nd\text{\_value}}][i,1]=0) then R[\sigma][i,1] := 0 else R[\sigma][i,1] := 100*\text{c}(\text{mu}[\sigma_{2nd\text{\_value}}][i,1]) fi od;

\text{table(} [ (3, 1) = .4109529366, (2, 1) = .7894581499, (1, 1) = .7955840609 ]\text{)\)

\(> \) the correlation coefficients and percentage of covariance of stress in the first- and second-order approximation

\(> \) Std[\sigma] := [seq(\text{cov}[\sigma_{2nd\text{\_value}}][i,i], i=1..n)]:

\(> \) \text{Std}_\text{dev}[\sigma] := \text{map(sqrt, Std[\sigma])}:

\(> \) for i from 1 to n do for j from 1 to n do if (\text{Std}_\text{dev}[\sigma][i]=0 or \text{Std}_\text{dev}[\sigma][j]=0) then \text{Rho}[i,j] := 0 else \text{Rho}[i,j] := \text{cov}[\sigma_{2nd\text{\_value}}][i,j]/(\text{Std}_\text{dev}[\sigma][i]*\text{Std}_\text{dev}[\sigma][j]) fi od od;

\text{matrix(n,n,'(i,j)->\text{Rho}[i,j]')}:

\(> \) C_temp := [seq(\text{mu}[\sigma_{2nd\text{\_value}}][i,1], i=1..n)]:

\(> \) B_temp := dotprod(C_temp, C_temp):

\text{cov}_\text{abs}[\sigma] := \text{map(abs, cov}[\sigma_{2nd\text{\_value}}]):

\(> \) \text{cov}_\text{percent}[\sigma] := \text{evalf}(100*\text{map(sqrt, evalm(cov}_\text{abs}[\sigma]/B_temp))):

\(> \) 4) \text{epsilon\_bar_value} := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf(epsilon\_bar)}, 12);

\(> \) \text{epsilon\_bar_value} := [.00209267823770, .00204050605458, -.000132172183129]

\(> \) \text{mu}[\epsilon_{1st\text{\_value}}] := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf(epsilon\_1st)}, 12);

\(> \) \text{mu}[\epsilon_{1st\text{\_value}}] := [.00209267823770, .00204050605458, -.000132172183129]

\(> \) \text{cov}[\epsilon_{1st\text{\_value}}] := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf(cov}[\epsilon_{1st\text{\_value}}]), 12);

\(> \) \text{cov}[\epsilon_{1st\text{\_value}}] :=
\begin{bmatrix}
.949781686554 e-7 & .835786179910e-7 & -.915091656170e-8 \\
.835786179910e-7 & .827034238209e-7 & -.405524301564e-8 \\
-.915091656170e-8 & -.405524301564e-8 & .621999059765e-8
\end{bmatrix}
\]

\(> \) \text{mu}[\epsilon_{2nd\text{\_value}}] := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf(epsilon\_2nd)}, 12);

\(> \) \text{mu}[\epsilon_{2nd\text{\_value}}] := [.00213066207022, .00207676001673, -.000132172183129]

\(> \) \text{cov}[\epsilon_{2nd\text{\_value}}] := \text{evalf}(['\text{sub}_1', \text{mu}[1]=1., \text{mu}[2]=1., \text{mu}[3]=2., \text{mu}[4]=30000.0, \text{mu}[5]=6.6e-6, \text{mu}[6]=50., \text{mu}[7]=100., \text{mu}[8]=100., \text{mu}[9]=0.1, \text{mu}[10]=0.15, l=100.], \text{evalf(cov}[\epsilon_{2nd\text{\_value}}]), 12);

\(> \) \text{cov}[\epsilon_{2nd\text{\_value}}] :=
\begin{bmatrix}
.949781686554 e-7 & .835786179910e-7 & -.915091656170e-8 \\
.835786179910e-7 & .827034238209e-7 & -.405524301564e-8 \\
-.915091656170e-8 & -.405524301564e-8 & .621999059765e-8
\end{bmatrix}
the ratio of stress vector in the second-order approximation

> delta[\mu[\epsilon_2nd_value]] := evalm(\mu[\epsilon_2nd_value]-\epsilon_bar_value):
> for i from 1 to n do if (\mu[\epsilon_2nd_value][i,1]=0) then  R[\epsilon][i,1] := 0 else R[\epsilon][i,1] :=
100*delta[\mu[\epsilon_2nd_value]][i,1]/\mu[\epsilon_2nd_value][i,1] fi od;
> print(R[\epsilon]);
table([ (3, 1) = 1.291892361,   (2, 1) = 1.745698189,   (1, 1) = 1.782724371 ])

the correlation coefficients and percentage of covariance of force in the first- and second-order approximation

> Std_\epsilon := \{seq(cov[\epsilon_2nd_value][i,i], i=1..n )\};
> Std_dev_\epsilon := map(sqrt,Std_\epsilon):
> for j from 1 to n do for i from 1 to n do  if (Std_dev_\epsilon[i]=0 or Std_dev_\epsilon[j]=0  ) then Rho[i,j] := 0 else Rho[i,j] := cov[\epsilon_2nd_value][i,j]/(Std_dev_\epsilon[i]*Std_dev_\epsilon[j]) fi od od;
> matrix(n,n,(i,j)->Rho[i,j]);
[1.000000000     .9430213935     -.3764938060]
[.9430213937     1.000000000     -.1787970200]
[-.3764938062    -.1787970201    1.000000000 ]
>C_temp := \{seq(\mu[\epsilon_2nd_value][i,1], i=1..n )\};
>B_temp := dotprod(C_temp, C_temp):
> cov_\epsilon_abs := map(abs,cov[\epsilon_2nd_value]):
> cov_\epsilon_percent := evalm(100*map(sqrt, evalm(cov_\epsilon_abs/B_temp)));

[10.34750915    9.706697964    3.211857148]
cov_\epsilon_percent := [9.706697964    9.655742335    2.138121428]
[3.211857148    2.138121428    2.648004919]
> cpu_time := (time()-settime)*seconds;
cpu_time := 103.299 seconds