TESSELLATIONS: LESSONS FOR EVERY AGE

A Thesis

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Master of Science

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ABSTRACT

Tessellations are a mathematical concept which many elementary teachers use for interdisciplinary lessons between math and art. Since the tilings are used by many artists and masons many of the lessons in circulation tend to focus primarily on the artistic part, while overlooking some of the deeper mathematical concepts such as symmetry and spatial sense. The inquiry-based lessons included in this paper utilize the subject of tessellations to lead students in developing a relationship between geometry, spatial sense, symmetry, and abstract algebra for older students. Lesson topics include fundamental principles of tessellations using regular polygons as well as those that can be made from irregular shapes, symmetry of polygons and tessellations, angle measurements of polygons, polyhedra, three-dimensional tessellations, and the wallpaper symmetry groups to which the regular tessellations belong. Background information is given prior to the lessons, so that teachers have adequate resources for teaching the concepts. The concluding chapter details results of testing at various age levels.
ACKNOWLEDGEMENTS

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I would also like to thank all of my professors and department staff, I have learned so much from you. Thank you Dr. Saliga for always making sure I kept things in perspective, and for taking time to laugh with me. Dr. Quesada, thank you for giving me the experience of teaching a computer based class. Dr. Clary for making learning fun and leading me to this research topic, even if you didn’t realize it. Dr. Wheland for your support and guidance through the past two years. Thanks also to Drs. Young and Clemons for introducing me to the program.

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Spatial thinking is a much needed skill in our world today. Some believe that it is an ability bestowed upon only a few lucky souls, though research during the past few decades indicates otherwise. The famous child psychologist Jean Piaget’s 1967 experiments lead him to the conclusion that spatial thinking is a cognitive process that can be developed through practice [4]. Since Piaget’s time there has been a strong foundation for development of spatial thinking. Recent research has also been done by Clements and Sarama [5] who state that

An adult’s ability to instantly “see” shapes in the world (their spatial sense) is the result, not the origin, of geometric knowledge. Young children learn a lot about shapes. *Geometry instruction needs to begin early.*

The 2000 revision of the *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (NCTM) [6] states repeatedly in it’s geometry standard that spatial reasoning is helpful in all aspects of everyday life such as using maps, designing floor plans, creating art and by learning to see the structure and symmetry around them. In the 1989 version of the NCTM standards Freudenthal is adamant about the importance of geometry and spatial sense stating
Geometry is grasping space...that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live breathe and move better in it.

Clements [7] notes that geometry and spatial reasoning are not only important for their own sake but they “form the foundation of much learning of mathematics and other subjects.” This is one of the critical aspects that must be kept in mind when working with students of any age.

Piaget as well as Clements believed that inquiry based approaches to learning geometry are the only effective methods of instruction. Children do not learn about geometry and space by passive observation but by interacting with shapes and their surrounding. Children must investigate concepts on their own to determine basic properties related to shapes and perspective. This investigation must be aided by a knowledgable supervisor, which is the basis of the inquiry approach.

An increased need for inquiry-based lessons in schools was a major topic discussed in the 2000 revision of the NCTM *Principles and Standards for School Mathematics* [6]. This publication stressed that ignoring the process standards, such as problem solving, reasoning, communication and connection can hinder student understanding of skills. So techniques used in inquiry-based lessons are necessary for student development.

Inquiry-based lessons are presented in Chapter 3 that are designed to develop geometry, spatial sense and symmetry concepts. Pierre van Hiele [8] believes that students’ ideas about geometry progress through various levels. For this reason the
lessons were developed to progress with increasing difficulty so that they can be used at various age levels, and lead the student to acquire mastery of the material by the end of their academic career. The lessons developed in this paper are compilations of previously developed concepts and original material in an attempt to merge existing lessons with inquiry-based techniques. All of the lessons are based on the topic of tessellations.

To properly give a lesson on tessellations, key areas must be included such as history, necessity, and fundamental reasoning. When these are instituted creatively and appropriately, a student will have the optimal chance of retaining the subject matter and fully developing the concepts portrayed. Tessellations is a topic that is not confined to a certain age range. Students in kindergarten through fourth grade can study patterns, how polygons fit together, lines of symmetry and naming regular tessellations by the number of sides each shape has. They can learn to draw their own tessellations and verify that the pattern can be repeated to cover an entire plane. Students at this age as well as the middle grades can begin to examine nonpolygonal tessellations such as those made by M. C. Escher and begin recognizing various methods of creating such tessellations. During the high school years, students can use angles of polygons to find tessellations, use angles to figure out which shapes fit together, and finally work towards proving theorems about tessellations.

While researching previous work done on the topic of tessellations it was found that many lessons focused on the relationship with symmetry and art but glossed over the inherent relationship to algebra. Articles such as Granger’s *Math*
is Art[9] discuss spending extensive time at the beginning of the school year showing students the beauty of math through geometry, mainly tessellations. He discusses various hands-on lessons to attempt to get students involved in creating art from mathematics but does not spend much time discussing the actual mathematics that he is trying to get across to the students. Others such as Chon-Keang [3] focused only on the geometric proofs, describing in detail how to determine each type of tessellation. Still others rely solely on computer software designed specifically for tessellations. The lessons contained in this paper strive to incorporate not only interdisciplinary aspects by showing math’s artistic side and appearance in nature and architecture but also how this topic can cross mathematical boundaries to integrate geometry, symmetry and algebra. The few lessons aided by the use of computer software use the geometry package Cabri II Plus, while the single computer based assessment uses standard Microsoft Paint or other accessible drawing software.

“Orchestrating classroom conversations so that the appropriate level of discourse and mathematical argumentation is maintained requires that teachers know mathematics well and have a clear sense of their mathematical goals for students” [6]. Consequently, extensive background information is included in Chapter 2 so that teachers will be able to properly execute the lessons found Chapter 3.
Tessellations have been used for centuries for artistic purposes as well as mathematical ones. They can be creative and beautiful such as the ones done by M. C. Escher or simply used to explain symmetry, rotation, and spatial relation of objects. A tessellation in basic terms is a tiling. The word “tessera” is derived from Latin and means a small stone cube. These cubes were used to make up “tessellata” which were the mosaic pictures forming floors and tilings in Roman buildings. Since then the term has become more specialized and is used to refer to pictures or tiles, mostly in the form of polygons, which cover the surface of a plane in a symmetrical way without overlapping or leaving gaps.

Basic knowledge of some geometric concepts is required to understand the majority of the material in this paper. Some terms and concepts which should be recalled follow:

1. Any n-sided polygon for integers $n \geq 3$ is the closed union of lines segments.

2. If a polygonal region is completely contained by the sides and interior of each of the polygons angles, it is said to be convex.
3. A regular polygon is a convex polygon having congruent sides and congruent angles.

4. The vertex of a polygon is the point where two sides meet. The vertex of a tessellation is the point where two or more polygon vertices meet.

5. The interior angle (or vertex angle) of a polygon is the angle formed by the joining of two edges and facing the interior of the polygon.

6. The interior angle sum of a convex polygon is $180(n - 2)$.

7. Each interior angle of a regular polygon has measure $\frac{180(n-2)}{n}$.

Teachers will need a background in Abstract Algebra to fully understand the sections dealing with wallpaper groups. Students need to be able to identify translations, rotations and reflections in complex patterns to benefit from the lessons on the wallpaper groups.

2.1 Regular Tessellations

A regular tessellation is a tiling of regular polygons which must follow three rules:

- all polygons must be the same type, so each vertex will look the same
- they must create a repeating pattern that covers the plane
- there must be no overlapping of shapes or gaps between shapes

The proof of why there are only three regular tessellations is rather simple and can be given to even young students. At least three regular polygons must surround a
vertex in order for it to tessellate the plane, for if we only have two polygons we have only created an edge rather than a vertex. Through experimentation we see that six equilateral triangles will encircle a vertex and four squares will do the same (this is a common tiling pattern on floors). Visually it can be seen that placing three regular pentagons around a vertex leaves a gap, while four pentagons will overlap. It should then be noted that three regular hexagons surround a vertex without leaving gaps or overlapping. A similar example can be made with regular heptagons, octagons, etc. to see that three of any regular polygon with more than six sides will overlap. Hence the only polygons that will form a regular tessellation are equilateral triangles, squares and regular hexagons.

To further illustrate the point for older students, who have had experience with interior angle measures, the following proof can be given. We can consider any vertex of a tessellation to be the center of a circle, so in order for an integer number of the same type of regular polygons to encircle a vertex the interior angle must divide 360°. We also know that the interior angle of any regular polygon is \((\frac{n-2)180°}{n}\), where \(n\) is the number of sides the polygon has. There are only three candidates for this as illustrated in the Table 2.1.

Note also that 360°/(interior angle) gives the number of polygons required to surround each vertex.

While the proof of why there are only three regular tessellations is quite simple, the proofs for determining the number of tessellations composed of more than one type of regular polygon or polygons which are not regular are more intricate and
2.2 Semiregular Tessellations

Semiregular tessellations use more than one type of regular polygon to tile the plane, while maintaining the restriction that each vertex must look the same. That is, the arrangement of the regular polygons around any vertex is the same. The arguments in this section are based on information found in Chong-Keang [3].

In order to determine the number of semiregular tessellations that exist we must look at several separate cases. Again, we know that at least three and no more than six polygons must surround a vertex. The only possibility where six regular polygons will surround a vertex is the regular tessellation (3, 3, 3, 3, 3, 3), so we need only consider the cases of three, four or five polygons around a vertex. Let us begin with the case of three polygons.

**Case 1** Suppose that only three regular polygons surround a vertex.
Let $n_1$, $n_2$ and $n_3$ be the number of sides of each polygon. It is known that the interior angle of any regular $n$-gon measures $\frac{180(n-2)}{n}$ and that $360^\circ$ surround any vertex so for three polygons we have

$$\frac{180(n_1 - 2)}{n_1} + \frac{180(n_2 - 2)}{n_2} + \frac{180(n_3 - 2)}{n_3} = 360$$

or

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2}$$

The possible solutions in Table 2.2 can be found through trial and error. Each solution describes a set of regular polygons which will surround a vertex with no overlapping or gaps, each of which is a potential semiregular tessellation arrangement. However, not all of these combinations can be extended in an infinite repeating pattern and so not all of these potential cases will work, which we will show in the following examples. Obviously the first case is a regular tessellation which was discussed earlier. To see why some of the other sets do not work, start by looking at those sets which contain a regular pentagon, namely $(5,5,10)$ and $(4,5,20)$. Each of these suggests a vertex arrangement in which a regular pentagon and two other polygons surround a vertex. Let one of the other polygons, call this polygon A, have

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>4</th>
<th>4</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_2$</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$n_3$</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>8</td>
<td>42</td>
<td>24</td>
<td>18</td>
<td>15</td>
</tr>
</tbody>
</table>
interior angle $\alpha$ and the third, polygon $B$, $\beta$. Figure 2.1(a) suggests that this type of arrangement cannot be extended to form a semiregular tessellation unless $\alpha = \beta$ or the shapes are the same. But this is not true for either case since $5 \neq 10$ and $4 \neq 20$. Hence neither of these sets generate a semiregular tessellation.

Figure 2.1: Certain configurations of three polygons do not form semiregular tessellations (a) Pentagons do not form semiregular tessellations (b) Equilateral triangles form semiregular tessellations when $\alpha = \beta$

A similar argument can be made for $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$ and $(3, 10, 15)$. With each of these we again have an odd sided figure along with two other shapes, which will require that the other two shapes be the same. See Figure 2.1(b) for reference. This argument can also be used to prove why $(3, 12, 12)$ will work since the other two shapes it tessellates with in fact are the same. To visualize the concept place two regular dodecagons edge to edge and then a third one below these so its
edges line up (Figure 2.2(a)). Observe the shape formed by the "gap" is an equilateral triangle. A similar visualization can be done with (4, 8, 8) (Figure 2.2(b)).

Figure 2.2: Examples of semiregular tessellations (a) Three dodecagons surround an equilateral triangle (b) Four octagons surround a square

The tessellation (4, 6, 12) in Figure 2.3 is different for we have three different types of shapes all with an even number of sides. We quickly see that the arrangement will work by placing squares and hexagons around the edges in an alternating fashion. The pattern can then be extended rotationally out from this original configuration. The key factor that allows this combination of shapes to work is the fact that each shape has an even number of sides and can hence be encircled by two other types of polygons in an alternating fashion. Thus we have they three of the possible ten arrangements which will produce a semi-regular tessellation as summarized in Table 2.3.

Case 2: Suppose that only four regular polygons surround a vertex.
Figure 2.3: The semiregular tessellation \((4, 6, 12)\)

Table 2.3: Sets of Three Polygons which Tessellate

<table>
<thead>
<tr>
<th>(n_1)</th>
<th>(n_2)</th>
<th>(n_3)</th>
<th>Tessellation?</th>
<th>(n_1)</th>
<th>(n_2)</th>
<th>(n_3)</th>
<th>Tessellation?</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
<td>Regular</td>
<td>3</td>
<td>7</td>
<td>42</td>
<td>No</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10</td>
<td>No</td>
<td>3</td>
<td>8</td>
<td>24</td>
<td>No</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>10</td>
<td>No</td>
<td>3</td>
<td>9</td>
<td>18</td>
<td>No</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
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<td>Semiregular</td>
<td>3</td>
<td>10</td>
<td>15</td>
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<tr>
<td>4</td>
<td>8</td>
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<td>Semiregular</td>
<td>3</td>
<td>12</td>
<td>12</td>
<td>Semiregular</td>
</tr>
</tbody>
</table>

Let \(n_1, n_2, n_3\) and \(n_4\) be the number of sides of a given polygon. Then

\[
\frac{180(n_1 - 2)}{n_1} + \frac{180(n_2 - 2)}{n_2} + \frac{180(n_3 - 2)}{n_3} + \frac{180(n_4 - 2)}{n_4} = 360
\]

or

\[
\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = 1
\]

This gives us the potential sets found in Table 2.4

The first set is the regular tessellation \((4, 4, 4, 4)\). Let us first look at a set that does form a semiregular tessellation. The set \((3, 4, 6, 4)\) when laid out as in Figure 2.4 will look the same at each vertex. Compare \(V_1\) and \(V_2\), if we focus on
Table 2.4: Possible Sets of Polygons for Semiregular Tessellations with Four Polygons

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>12</th>
<th>12</th>
<th>6</th>
<th>6</th>
<th>6</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>n₁</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>n₂</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>n₃</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>n₄</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the triangle at V₁ and work in a clockwise fashion we have a square, hexagon and square. Notice the same arrangement around V₂. Now look at (3, 6, 3, 6) in Figure 2.4(a) 2.4(b).

Figure 2.4: Semiregular tessellations (a) (3, 4, 6, 4) (b) (3, 6, 3, 6)

2.4(b). This will produce a semiregular tessellation which resembles a star pattern. Look at each vertex, note again that the shapes surround each vertex in the same order. This brings us to the topic of why (3, 6, 3, 6) works while (3, 3, 6, 6) does not. If we try to arrange the shapes in the order 3, 3, 6, 6 at V and V₁ as seen in Figure 2.5(b) we see that we will need three triangles at V₂, which is too many. However, if we instead change the arrangement at V to 3, 3, 6, 6 we have what is known as a
“demiregular” tessellation (Figure 2.5 (b)). The definition and number of defined demiregular tessellations differs greatly by author but for the purpose of this paper we will refer to a demiregular tessellation as one which has two uniform vertices, in this case $V_1$ is $(3,6,3,6)$ and $V_2$ is $(3,3,6,6)$. Although this is not a semiregular tessellation it is mentioned in this section because it is a pattern often confused as one and found by students when completing the activities found in Chapter 3.

Figure 2.5: Attempts to tessellate with $(3,3,6,6)$ (a) The initial configuration does not work (b) Example of a Demiregular Tessellation with Triangles and Hexagons

Now let us look at why the remaining do not generate semiregular tessellations. Consider $(3,3,4,12)$ as seen in Figure 2.6.

We begin with our initial configuration around vertex $V$. To extend this pattern by building the same arrangement around $V_1$, it is impossible to have the proper layout for $V_2$. Hence $(3,3,4,12)$ is not a semiregular tessellation.

A similar proof can be done to show the remaining sets $(3,4,3,12)$ (Figure 2.7(a)) and $(3,4,4,6)$ (Figure 2.7(b)) are not semiregular tessellations. A summary
Case 3: Suppose that five regular polygons surround a vertex.

This gives us the potential sets found in Table 2.6, all of which form semiregular tessellations. We have now found the eight particular arrangements of regular polygons that generate our semiregular tessellations.

Figure 2.6: The set of polygons (3, 3, 12, 4) does not form a tessellation

Figure 2.7: Certain configurations of four polygons do not form semiregular tessellations (a) (3, 4, 3, 12) (b) (3, 4, 4, 6) [3]

of the tessellations with four polygons can be found in Table 2.5.
Table 2.5: Sets of Four Polygons which Tessellate

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>Tessellation?</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>Regular</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>12</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>Demiregular</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>Semiregular</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>Semiregular</td>
</tr>
</tbody>
</table>

Table 2.6: Possible Sets of Polygons for Semiregular Tessellations with Five Polygons

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$n_5$</th>
<th>Tessellation?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>Semiregular</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>Semiregular</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>Semiregular</td>
</tr>
</tbody>
</table>

2.3 Tessellations with Nonregular Polygons

Now that we know how we can use regular polygons to tessellate the plane, the next question is “which nonregular polygons tessellate the plane?” We start by proving that all parallelograms tessellate. We then determine that all triangles and all quadrilaterals tessellate. Finally, we examine some pentagons and hexagons which
tile the plane.

To show that any parallelogram will tessellate we recall that a parallelogram is a quadrilateral with opposite sides parallel (and therefore opposite angles equal). We also know that the sum of the interior angles of a parallelogram is 360°, which is what we would need to surround a vertex for tessellation. So if we translate the shape to the left, right, top and bottom infinitely we see that we get a tessellation that resembles a skewed grid (Figure 2.3). Thus, all parallelograms tessellate. Since any triangle paired with another congruent triangle (a copy of itself) forms a parallelogram (Figure 2.9(a)), we get that all triangles tessellate (Figure 2.9(b)).

To determine that all quadrilaterals tessellate we begin by noting that for all quadrilaterals the sum of its interior angles is 360°. Consequently, we can place four of these shapes with one of each angle around a vertex (Figure 2.10(a)). This

3 polygons: (4, 6, 12) (4, 8, 8) (3, 12, 12)
4 polygons: (3, 6, 3, 6) (3, 4, 6, 4)
5 polygons: (3, 3, 3, 6) (3, 3, 3, 4, 4) (3, 3, 4, 3, 4)
Figure 2.9: Pictoral proof that all triangles tessellate 
(a) Two triangles create a parallelogram (b) Tessellation of scalene triangles as a parallelogram is easily visualized for convex quadrilaterals (Figure 2.10(a)) but a little trickier to visualize with nonconvex (Figure 2.10(b)) because they appear to have jutting pieces which could obstruct their ability to fit four shapes around a vertex. Now this group of shapes will tessellate because we can translate it so that each vertex will have the shapes surrounding it in the same order. So all quadrilaterals tessellate.

Figure 2.10: Tessellations of Quadrilaterals (a) Convex and (b) Nonconvex

Knowing that all triangles and quadrilaterals tessellates the question may then occur, “Do all pentagons tessellate?” We have shown that regular pentagons
do not tessellate but are there some which will? Figure 2.11(a) shows a pentagon with all equal sides but different angles that tessellates. Notice that each pentagon if formed by attaching an equilateral triangle and square together at their bases. In fact any polygon formed by the merging of a triangle and rectangle with congruent bases yields a tessellating pentagon. Another unique feature of this tessellation is that combining two of these pentagons produces an odd shaped hexagon which will also tessellate (Figure 2.11(b)). Other examples of tessellating pentagons which form tessellating hexagons are shown in Figures 2.12.

![2.11(a)](image)

![2.11(b)](image)

Figure 2.11: Tessellations of nonregular polygons (a) Pentagons (b) Hexagons

2.4 Polyhedra

In geometry, a polyhedron is a three-dimensional solid which consists of a collection of polygons joined at their edges. The word derives from the Greek poly (many) plus the Indo-European hedron (seat) [10]. The plural of polyhedron is “polyhedra” (or
Tessellations and regular polyhedra are closely related, and rely on similar geometric principles. It has been discussed that in order for a set of polygons to create a tessellation the sum of their interior angles must equal 360°. To form a regular polyhedron the sum of the interior angles must be less than 360° so that the polygons have space to “fold up”.

As with tessellations we noted that there are two special types known as regular and semiregular. There is a similar distinction with polyhedra. When we examined regular tessellations, as well as regular polyhedra, we required that only one type of polygon be used. With semiregular tessellations we used more than one type of polygon with the restriction that each vertex look the same. Similarly, a semiregular polyhedron is constructed from more than one type of polygon and each vertex is the same. There are 14 semiregular polyhedra, which have been known for centuries.
2.4.1 Regular Polyhedra

There are only five polyhedra with regular convex polygonal faces, which we refer to as convex regular polyhedra. The proof of why there are only five of these is fairly simple and can be explained to students who can manipulate inequalities and factor by grouping. We begin by examining a vertex of a polyhedra. If there are $m$ regular $n$-gons (polygons with $n$ sides) surrounding a vertex we need $m(180^\circ(n-2)/n) < 360^\circ$. To get a simpler relationship between $m$ and $n$ observe:

\[
m(180^\circ(n-2)/n) < 360^\circ
\]
\[
m(n-2) < 2n
\]
\[
mn - 2m - 2n < 0
\]
\[
mn - 2m - 2n + 4 < 4
\]
\[
(n - 2)(m - 2) < 4
\]

Lastly, we know that every polygon has at least 3 sides so $n > 2$. So the only integer pairs $(n, m)$ that satisfy the inequality $(n - 2)(m - 2) < 4$ are $(3, 3), (4, 3), (3, 4), (5, 3)$ and $(5, 3)$, elaborated upon in Table 2.7.

Figure 2.13 depicts the tetrahedron or regular triangular pyramid, regular square prism or cube, the octahedron, the dodecahedron and the icosahedron in order from left to right.
Table 2.7: Regular Polyhedra Information

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>((n-2)(m-2))</th>
<th>Name</th>
<th>Number of Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>Tetrahedron</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>Cube</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>Octahedron</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>Dodecahedron</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3</td>
<td>Icosahedron</td>
<td>20</td>
</tr>
</tbody>
</table>

Figure 2.13: The Five Regular Polyhedra (left to right) Tetrahedron, Cube (square prism), Octahedron, Dodecahedron, Icosahedron

Although the neolithic people of Scotland developed the five solids a thousand years earlier [11], these polyhedra are often referred to as the Platonic solids because they were known to the ancient Greeks, and were described by Plato in his Timaeus. Plato equated the tetrahedron with the “element” fire, the cube with earth, the icosahedron with water, the octahedron with air, and the dodecahedron with the stuff of which the heavens were made [10].
2.4.2 Semiregular Polyhedra

One way of determining some of the semiregular polyhedra looks at the semiregular tessellations and decreases the number of sides of one of the polygons by one or two. This gives the shapes the space necessary to “fold up.” For example, if we change the hexagon in the semiregular tessellation \( (3, 3, 3, 3, 6) \) to a pentagon and attach the figures so as to not leave any gaps we create what is referred to as a snub dodecahedron (Figure 2.14).

Figure 2.14: Adaptations of semiregular tessellations create semiregular polyhedra (a) \( (3, 3, 3, 3, 6) \) creates the (b) The snub dodecahedron [1]

Similarly, if we change the dodecagon in \( (12, 4, 6) \) to a decagon we have the great rhombicosidodecahedron as seen in Figure 2.15. One of the advantages to this approach of finding semiregular polyhedra is that we know that the vertices of the semiregular tessellations are all the same so changing only one type of shape will maintain this necessary requirement for semiregular polyhedra.

Another unique fact about the regular polyhedra is the concept of duality. If we look at the center point of each face on a regular polyhedron and connect these
Figure 2.15: Adaptations of semiregular tessellations create semiregular polyhedra (a) $(12, 4, 6)$ creates (b) The great rhombicosidodecahedron [1]

points with all neighboring points on adjacent faces we will get one of the other regular polyhedra. For example, looking at the cube we see that these lines produce an octahedron (see Figure 2.16). This phenomenon occurs when the number of faces of one polyhedron equals the number of vertices of another. Looking at Table 2.8 we see that there are three "pairings" of the regular polyhedra which form duals; the cube and the octahedron have already been discussed, the dodecahedron and the icosahedron, and the tetrahedron which is it’s own dual or self-dual. Notice that the number of edges will be the same for both polyhedra in each pair. We will now see that we can use a concept based on duals to find some of the semiregular polyhedra.

Many of the semiregular polyhedra can be formed by examination of the duals of the regular polyhedra and through a process called truncation. Truncation is the act of “slicing” or “shaving” off part of the solid, usually the vertices. Looking at the cube and its dual the octahedron we can find three semiregular polyhedra through this process. Starting with the cube and truncating each corner until we are left
with an equilateral triangle and the six faces, which were formerly squares, become octagons creating the new polyhedra called simply the truncated cube. Similarly, if we truncate each vertex of an octahedron so that the original faces become hexagons and the vertices yield squares, we produce the truncated octahedron. If we continue truncating these vertices until we are left with large equilateral triangles and square faces on the diagonal of the original faces we arrive at the cube octahedron.

The cube octahedron can also be found by bisecting each edge of either the cube or
Figure 2.17: Truncated regular polyhedra form semiregular polyhedra
(a) The Truncated Cube (b) The Truncated Octahedron [1]

the octahedron, connecting the points and truncating the vertices (Figure 2.18).

Figure 2.18: The cube octahedron [1]

If we use this process on the dodecahedron and the icosahedron we get the intermediate semicircular polyhedra: the truncated dodecahedron, the icosidodecahedron and the truncated icosahedron, which resembles a soccer ball (Figure 2.19).

Since the tetrahedron is a self-dual we only get one new polyhedra from this slicing method, namely the truncated tetrahedron in Figure 2.20. This is formed by trisecting the edges of the tetrahedron, connecting the points and slicing off the corners.
As mentioned, there are 14 semiregular polyhedra, which can be found using truncation or manipulation of the semiregular tessellations. Many of them can actually be found using either method. Take for example the truncated dodecahedron which can obviously found by truncating the dodecahedron. It can also be found by changing the dodecahedra in the semiregular tessellation (12, 3, 12) to decagons and folding the shapes together to remove the gaps (Figure 2.21).

A summary of all 14 can be found in Table 2.9. The name, number of faces, vertices and edges as well as the number of each type of polygon required to build the polyhedra are listed.
Figure 2.21: Adaptations of semiregular tessellations create semiregular polyhedra (a) \((12, 3, 12)\) creates (b) The truncated dodecahedron [1]

2.5 Wallpaper Groups

Almost everyone at some point in their lives has observed a wallpaper pattern. The name itself gives a brief explanation of what the set contains, some type of repeating pattern, which we use to decorate our walls. Although this is true it is only one of the many uses for what mathematicians refer to as the wallpaper groups, symmetry groups or planar crystallographic groups. All of the tessellations we have just discussed are examples of certain wallpaper groups. Teachers will need a background in Abstract Algebra to fully understand the sections dealing with wallpaper groups. Students need to be able to identify translations, rotations and reflections in complex patterns to benefit from the lessons on the wallpaper groups. Before giving a full description of what this group includes, a few definitions must be given.

When we look at these beautiful patterns we can examine the exact way in
which they repeat. In most we will observe translation or shifting from one place to another along a straight line. In others we may observe rotation about a point or reflection about a point or a line. These are all what mathematicians refer to as isometries. According to Johnston and Richman, an isometry is a linear transformation of the plane (or space) that preserves distances between points [12]. The compositions of the isometries listed above are isometries themselves. For example, glide reflection is a simultaneous translation and reflection in the center line. A common example used to illustrate the concept of glide reflection is the pattern which footprints create.

Now let us take a look at the wallpaper groups. Each group is comprised of a set of patterns, each adhering to the same set of isometries. It has been shown that there are only 17 classes of wallpaper groups; all others are isomorphic to these. The full mathematical explanation is subtle and involved, but Schattschneider [13] gives a few key concepts that the multi-page proof deals with.

The main point is that any group of symmetries that is big enough to have the wallpapering property, is big enough, that is, has a single tile for its fundamental domain. . . . there are only a few choices for tile shapes that will actually fit together and cover the plane, like a rectangle, an equilateral triangle, and so on.

In other words each wallpaper pattern can be created by a single, specifically shaped tile translated infinitely in all directions. However, not all repeating patterns are
wallpaper patterns. One of the key factors in determining if a pattern is really a wallpaper pattern is that it must be periodic. This means that a lattice can be constructed to hold identical pieces of the pattern. A lattice is a grid consisting of two sets of parallel (not necessarily vertical and horizontal), evenly spaced lines. An example to visualize this concept would be a chain link fence. This grid will form parallelograms, which are called period parallelograms. Despite the fact that each grid pattern is a parallelogram, there are five possible shapes which can be formed by these grids based on their symmetry. All parallelograms have rotational symmetry, if they also have two reflection symmetries it is rectangular, rhombic or hexagonal. These three are distinguished as follows: rectangular grids have reflections through the sides, rhombic and hexagonal through the vertices or grid points but hexagonal have interior angles of $60^\circ$ and $120^\circ$ in order for the grid unit to form two equilateral triangles (Figure 2.22). Lastly, the square will have four reflection symmetries.

The elements of the group are identical pieces of the tiling that fit in each of these parallelogram, not necessarily the whole parallelogram. The operations on each element depends on the group to which it belongs. These elements are called fundamental domains (or fundamental units) for the tiling. Once you know you have a short list of fundamental domain shapes, all that is left to do is figure out how many ways you can put them together. This is mostly a matter of listing what kinds of symmetries you can use to glue together fundamental domains, and seeing how many ways you can combine them. Some sources go as far as describing a translation region. This is the smallest area that can be formed from a fundamental
domain through rotations and reflections so that the remainder of the pattern can be generated by using only the translations of the group created in this translation region. For the purposes of this paper I will focus on only the 11 groups to which the regular and semiregular tessellations belong as well as a few common real world occurrences of wallpaper patterns. I will also discuss how coloring these tessellations in a certain fashion can cause the tessellation to possess different isometries than if it were not colored and hence belong to a different wallpaper pattern group. All of the groups are summarized in Table 2.10 adapted from [2].

Lastly, we must discuss notation. For this paper I will be using crystallographic or Hermann-Mauguin-like notation to identify each translation region. This notation uses four symbols identifying the conventionally chosen fundamental domain, the highest order of rotation and other symmetries fundamental to the group.
In a “primitive cell”, denoted by \( p \), the lattice described earlier is constructed such that the vertex points of each lattice unit are points of highest order rotation. This is the basis for all but two of the wallpaper groups. The remaining two groups are based on “centered cells”, denoted with a \( c \). These centered cells are chosen so that reflection axes will be perpendicular to one or both sides of the cell. When reading the name of a wallpaper group from left to right the first position is set by either a \( p \) or a \( c \) for primitive or centered cell followed by an integer for the highest order of rotation. \(^1\) The third symbol denotes a symmetry axis such as \( m \) for mirror lines or lines of reflection, \( g \) for glide-reflection\(^1\) axis but no lines or reflection and a 1 indicates no symmetry axis. If there is a number in the forth position the group has a symmetry axis not perpendicular to one of the sides of the cell. If there are no symbols in the third or forth position then the group contains no reflection symmetries \([13]\). For example the group \( p6m \) has six rotations around a center point as well as reflections in six directions. The website designed by Geometry Technologies \([14]\) has an excellent Java applet to show the exact motions of all of the fundamental domains or each wallpaper group in completing a tiling. With this said we will now describe the characteristics of each of these classes.

\(^1\)A rotation through an angle of \( 360^\circ/n \) is said to have order \( n \).

\(^1\)A glide-reflection is nontrivial if its component translation and reflection are not symmetries of the pattern.
2.5.1 Translation and One Isometry

The first, and simplest, group is \( p_1 \). In this group the fundamental domain translates along the lattice and hence has only one rotation of 360°. In other words it slides along the parallel lines created by the lattice in two directions. All symmetry groups possess translation on their fundamental domain as an isometry, but \( p_1 \) has only this action. The lattice for this group is “parallelogrammatic” or in the form of a parallelogram. Observe how the colored semiregular tessellation(3, 3, 3, 4, 4) has this translation only pattern (Figure 2.23(a)). The next four wallpaper groups discussed are based on \( p_1 \) with the addition of only one other isometry.

![Figure 2.23: Examples of wallpaper patterns](image)

Figure 2.23: Examples of wallpaper patterns (a) The colored semiregular tessellation (3, 3, 3, 4, 4) is a \( p_1 \) pattern (b) Chain link fence produces a \( p_2 \) pattern

If we take the fundamental domain of \( p_1 \) and incorporate a rotation of \( a = 180°, 120°, 90° \) or \( 60° \), then we have the symmetry groups \( p_2, p_3, p_4 \) and \( p_6 \)
respectively. To create a pattern in each of the $pk$ groups only requires translations and rotations of degree $a = 360°/k$, where $k$ is the number of rotations denoted by the pattern name (2, 3, 4 or 6).

Students often identify the pattern in a chain link fence (Figure 2.23(b)) as (4, 4, 4, 4). If we look carefully though, we see that the twists in the wire yields only two rotations and no reflections which is exactly what an element (translation region) of $p2$ needs.

An example of $p3$ would be the colored version of the semiregular tessellation (3, 3, 3, 3, 6) found in Figure 2.24(a). We first note that this specific coloring has rotations of degree three centered at the center of each hexagon. If we remove the coloring and just focus on the black and white outline (Figure 2.24(b)) we see that we now have six rotations with the same center of rotation as before but 6 rotations of tells us that the new pattern is $p6$. Note also that this pattern contains no lines of reflection regardless of coloring.

If we look at the colored version of the semiregular tessellation (3, 3, 4, 3, 4) (Figure 2.5.1) we see that it’s only isometry is a rotation of 90° centered at points A and B (there is also a 180° rotation centered at point C.) This isometry tells us that this pattern belongs to the $p4$ group. We will see in the next section how removing the color from this pattern yields a pattern with translation plus two isometries.
2.24(a)  

2.24(b)  

Figure 2.24: Examples of wallpaper patterns (a) The semiregular tessellation (3, 3, 3, 3, 6) with this coloring belongs to the $p3$ group.  
(b) The semiregular tessellation (3, 3, 3, 3, 6) with no coloring belongs to the $p6$ group.  

Figure 2.25: The colored tessellation (3, 3, 4, 3, 4) belongs to the $p4$ wallpaper group.

2.5.2 Translation with Two Isometries

We now look at groups that contain translations plus two other isometries. The first group we will look at is $pmm$ whose two isometries are both reflection as denoted by the two $m$’s in its name. These reflections are over perpendicular axes forming a rectangular fundamental region. In other words the rectangle is reflected across each or its sides to form the translation region. Its translation region can be created by
taking any four rectangles surrounding a vertex. Figure 2.26(a) depicts the coloring of the regular tessellation \((4, 4, 4, 4)\) which is a \(pmm\) wallpaper pattern. Although \((4, 4, 4, 4)\) uses a square polygon to create its pattern, the different color of each square surrounding a vertex gives it the distinguishing characteristic of the \(pmm\) group.

![2.26(a)](image1)

![2.26(b)](image2)

Figure 2.26: Examples of wallpaper patterns (a) The regular tessellation \((4, 4, 4, 4)\) with this coloring is a \(pmm\) pattern

(b) The checkerboard coloring is a \(p4m\) pattern

If we remove the coloring of \((4, 4, 4, 4)\) or even replace it with a checkerboard pattern we see that two rotations as well as two diagonal reflections are added for a total of four of each. This causes the pattern to now belong to the \(p4m\) group. Note that this pattern also has a nontrivial glide reflection. Another example of a \(p4m\) pattern is the uncolored semiregular tessellation \((4, 8, 8)\) (Figure 2.27(a)). Notice the 90° rotations centered at the center of each octagon or square with reflections in the horizontal, vertical and both diagonal directions.
Figure 2.27: Examples of wallpaper patterns (a) The semiregular tessellation (4.8.8) uncolored is a \( p4m \) pattern
(b) (3, 3, 4, 3, 4) is a \( p4g \) pattern

An example of \( p4g \) is seen when we remove all color from (3, 3, 4, 3, 4). Notice now that the center of each square is a center of rotation of degree four but that there are only two lines of reflection which bisect the triangles. So there are four rotations but only two reflections and a glide reflection which are the properties of group \( p4g \).

We revisit the tessellation (3, 3, 3, 4, 4) to discuss an example of \( cmm \). If we look at the pattern uncolored we see that there are reflections in the horizontal and vertical directions and that it has 180° rotations centered at repeating points A and B. This group is distinguished from \( pmm \) by the fact that the center of rotation B is not centered on the reflection axes although it is on a glide-reflection axis, see Figure 2.28(a). This follows with the wallpaper pattern group \( cmm \). Another example of the \( cmm \) group is the brickwork pattern often used for building and paving (Figure 2.28(b))

I conclude with the group patterns of \( p6m \) and \( p3m1 \), both of which encom-
2.28(a) 2.28(b)

Figure 2.28: Examples of wallpaper patterns
(a) The semiregular tessellation $(3, 3, 3, 4, 4)$ uncolored belongs to the $cmm$ pattern group
(b) The brickwork pattern also belongs to the $cmm$ pattern group

pass over half of the regular and semiregular tessellations. If we look at the remaining tessellations, $(3, 3, 3, 3, 3)$, $(6, 6, 6)$, $(3, 12, 12)$, $(3, 6, 3, 6)$, $(4, 6, 4, 12)$ and $(3, 4, 6, 4)$, uncolored we see that each has $60^\circ$ rotation as well as six different directions of reflection axes and nontrivial glide reflections. All fit the description of the $p6m$ group pattern. Now if we color these patterns as seen in Figure 2.29 we notice that there are only $120^\circ$ rotation symmetries with three directions of reflection and nontrivial glide reflections. The centers of rotation lie on the reflection axes, so this puts all six in the group $p3m1$. If a pattern has other rotation centers not on reflection axes, then it would be in the group $p31m$. An example of the is shown in Figure 2.30.

Although it may have originally seemed that these groups only had decorative or mathematical purposes, we now know they are useful for much more. One of the naming schemes for the wallpaper groups was actually derived from the crystalline
structures that produce the lattice grids on which these patterns are built. Previous work with symmetry groups had been done by mathematicians for centuries, but
many of the applications were not realized until the 1900s. At this time two British scientists, Sir Bragg and his son Lawrence, deduced that the pattern diffracted by the unit cell, the fundamental domain, of a crystal would produce these lattice structures upon which symmetry groups are built, and hence the entire structure of the crystal could be determined by the actions of the isometries (Figure 2.31(a)). In X-ray crystallography, a crystallized molecule is bombarded with a beam of X-rays. Upon reaching the crystallized structure, the rays diffract according to the electron density of the atoms. Thus the more dense the electrons, the greater the rays diffract. This diffraction pattern is then recorded on photographic film (Figure 2.31(b)). Deciphering the diffraction pattern can take months or even years, but pays off in the end when the final results are obtained [2].

For example the technique of X-ray crystallography was used by Rosalind Franklin revealing that the DNA molecule had a "helical shape" with repeating distances of 0.34 nm, 2nm and 3.4 nm. She was unsure though as to whether it was a double or triple helix. Watson and Crick later determined that the DNA molecule was a double helix comprised of nucleotides which were complementary base paired.

X-ray crystallography is also a key element used to draw conclusions about germ cells and mature antibodies. The process is applied to the specified cells and a diffraction pattern is recorded. A scientist must then decipher the diffraction pattern generated with the aid of computers and knowledge of proteins, amino acids, and DNA. These examples give proof to the importance of theoretical mathematics in application and science.
Figure 2.31: X-ray Images form lattice structures (a) Diffraction image produced by a crystal (b) Crystallography image of DNA
Table 2.9: Useful Information for Thirteen Semiregular Polyhedra [1]

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Triangles</th>
<th>Squares</th>
<th>Pentagons</th>
<th>Hexagons</th>
<th>Octagons</th>
<th>Decagons</th>
<th>V</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Hexahedron</td>
<td>-</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>8</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Octahedron</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20</td>
<td>30</td>
<td>12</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>Tr. Tetra.</td>
<td>4</td>
<td>-</td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>18</td>
<td>8</td>
</tr>
<tr>
<td>Tr. Cube</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>Tr. Octahedron</td>
<td>-</td>
<td>6</td>
<td>-</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>Tr. Dodeca.</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>Tr. Icosahedron</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>60</td>
<td>90</td>
<td>32</td>
</tr>
<tr>
<td>Cube Octahedron</td>
<td>8</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>12</td>
<td>24</td>
<td>14</td>
</tr>
<tr>
<td>Rhombicubocta.</td>
<td>8</td>
<td>18</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>24</td>
<td>48</td>
<td>26</td>
</tr>
<tr>
<td>Tr. Cubocta.</td>
<td>-</td>
<td>12</td>
<td>-</td>
<td>8</td>
<td>6</td>
<td>-</td>
<td>48</td>
<td>72</td>
<td>26</td>
</tr>
<tr>
<td>Snum Cube</td>
<td>32</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>24</td>
<td>60</td>
<td>38</td>
</tr>
<tr>
<td>Icosidodeca.</td>
<td>20</td>
<td>-</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>30</td>
<td>60</td>
<td>32</td>
</tr>
<tr>
<td>Rhombicosidodeca.</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>60</td>
<td>120</td>
<td>62</td>
</tr>
<tr>
<td>Tr. Icosidodeca.</td>
<td>-</td>
<td>30</td>
<td>-</td>
<td>20</td>
<td>-</td>
<td>12</td>
<td>120</td>
<td>180</td>
<td>62</td>
</tr>
<tr>
<td>Snum Dodecahedron</td>
<td>80</td>
<td>-</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>60</td>
<td>150</td>
<td>92</td>
</tr>
</tbody>
</table>
Table 2.10: Identification Chart for Wallpaper Patterns [2]

<table>
<thead>
<tr>
<th>Type</th>
<th>Lattice</th>
<th>Highest Order of Rotation</th>
<th>Reflections</th>
<th>Nontrivial Glide Reflection</th>
<th>Generating Region</th>
<th>Helpful Distinguishing Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>Parallelogram</td>
<td>1</td>
<td>No</td>
<td>No</td>
<td>1 unit</td>
<td></td>
</tr>
<tr>
<td>$p_2$</td>
<td>Parallelogram</td>
<td>2</td>
<td>No</td>
<td>No</td>
<td>$\frac{1}{2}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_3$</td>
<td>Hexagonal</td>
<td>3</td>
<td>No</td>
<td>No</td>
<td>$\frac{1}{3}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_4$</td>
<td>Square</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>$\frac{1}{4}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_6$</td>
<td>Hexagonal</td>
<td>6</td>
<td>No</td>
<td>No</td>
<td>$\frac{1}{6}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_{mm}$</td>
<td>Rectangular</td>
<td>2</td>
<td>Yes</td>
<td>No</td>
<td>$\frac{1}{4}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_{4m}$</td>
<td>Square</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>$\frac{1}{8}$ unit</td>
<td>4-fold center on ref axes</td>
</tr>
<tr>
<td>$p_{4g}$</td>
<td>Square</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>$\frac{1}{8}$ unit</td>
<td>4-fold center not on ref axes</td>
</tr>
<tr>
<td>$c_{mm}$</td>
<td>Rhombic</td>
<td>2</td>
<td>Yes</td>
<td>Yes</td>
<td>$\frac{1}{4}$ unit</td>
<td>⊥ reflection axes</td>
</tr>
<tr>
<td>$p_{6m}$</td>
<td>Hexagonal</td>
<td>6</td>
<td>Yes</td>
<td>Yes</td>
<td>$\frac{1}{12}$ unit</td>
<td></td>
</tr>
<tr>
<td>$p_{3m1}$</td>
<td>Hexagonal</td>
<td>3</td>
<td>Yes</td>
<td>Yes</td>
<td>$\frac{1}{6}$ unit</td>
<td>All 3-fold centers on ref axes</td>
</tr>
</tbody>
</table>

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3.1 Lessons and Activities

3.1.1 Polygon Reference Sheet

Use the following formulas to fill in the chart below and determine the necessary angles for each polygon. These formulas should follow from their definitions. Note: These formulas can only be used for regular polygons. The equilateral triangle is filled in as an example.

$n$ – the number of sides a polygon has

Central Angle $= \frac{360^\circ}{n}$

Vertex Angle $= \frac{180^\circ(n-2)}{n}$

Exterior Angle $= 180^\circ$ – vertex angle

In other words Vertex Angle + Exterior Angle $= 180^\circ$ since they are supplementary
<table>
<thead>
<tr>
<th>Name of Polygon</th>
<th>Number of Sides</th>
<th>Central Angle</th>
<th>Vertex Angle</th>
<th>Exterior Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral Triangle</td>
<td>3</td>
<td>120°</td>
<td>60°</td>
<td>120°</td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Pentagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Hexagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Heptagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Octagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Nonagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Decagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Hendecagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Dodecagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do you notice any relationship between the Exterior Angle and the Central Angle of regular polygons?

Can you explain why this relationship occurs?
3.1.2 Lesson One: Finding Tessellations and Tilings

**Learning Objective:** Create regular polygon tessellations using hands on manipulatives.

*What is a Tessellation?* A tessellation in basic terms is a tiling. The word “tessera” is derived from Latin and means a small stone cube. These cubes were used to make up “tessellata” - the mosaic pictures forming floors and tilings in Roman buildings. Since then, the term has become more specialized and is used to refer to pictures or tiles, mostly in the form of polygons, which cover the surface of a plane in a symmetrical way without overlapping or leaving gaps.

*Why are tessellations important?* Tessellations have been used for centuries for artistic purposes as well as mathematical ones. They can be creative and beautiful such as the ones done by M. C. Escher or simply used to explain symmetry, rotation, and spatial relation of objects.

![Figure 3.1: Alhambra Tiling](image1) ![Figure 3.2: M.C. Escher’s 8 Heads](image2)
Tessellations can even be as simple as what you see on the floor. Look at the floor of your classroom. What shape do you see?

- Does this shape cover the entire floor without any gaps or overlapping?

Now look at the walls.

- Is this the same tiling as the floor?

Now let’s make some tessellations of our own!

_Tessellations_

The Rules: There are a few variations as to the definition of a tessellation, but the first rule is always an essential part of the description.

**Rule 1:** The Tessellation must tile the floor (if it were to go on forever) with no overlapping of shapes or gaps.

**Rule 2:** The tiles must be regular polygons.

1. Design some of your own tessellations using the given shapes and following the two rules above. Trace them onto a piece of paper and fill in each shape with its corresponding color. There are many different designs that can be made. See how many different tessellations you can make. You can use 1, 2, 3 or 4 different shapes to make each tessellation. Look at the table from the handout on polygons. See if this gives you any hints as to which shapes “fit together.” Try to make 10 – 12 different tessellations. How many tessellations did your group create? What shape combinations did you make?
2. Now that you’ve found about a dozen different tessellations place them into 3 or 4 categories. The categories can be based on whatever criterion you wish. The categories can have intersecting regions or they can be disjoint. Make a list of the categories your group came up with and how many tessellations were in each

3. Discuss with another group what you’ve found. Did you both find the same kinds of tessellations? Did your group find some that the others did not? Did you both come up with the same categories of classification?

3.1.3 Lesson Two: Naming Tessellations

**Learning Objective:** Learn naming scheme for regular and semiregular tessellations and categorize different tessellations into regular, semiregular or neither.

*Three Categories of Tessellations*

Below are the categories that most mathematicians use when describing tessellations and the rules for each.

*Regular Tessellations*

**Rule 1:** The Tessellation must tile the plane (if it were to go on forever) with no overlapping of shapes or gaps.

**Rule 2:** The tiles must be regular polygons and all the same shape.

**Rule 3:** Each vertex must look the same.
Did your group find any of these? If so how many? Which ones?

Semi-regular Tessellations

Semi-regular Tessellations follow some of the same rule as Regular Tessellations but can use more than one shape. These are sometimes called the Archimedean Tessellations.

Rule 1: The Tessellation must tile the plane (if it were to go on forever) with no overlapping of shapes or gaps.

Rule 2: The tiles must be regular polygons.

Rule 3: Each vertex must look the same.

Did your group find any of these? If so how many? Which ones?
NOT Semi-regular Tessellations

What happens if we use regular polygons but they don’t have the same shapes at each vertex? These are what mathematicians call not-semi-regular tessellations.

**Rule 1:** The Tessellation must tile the plane (if it were to go on forever) with no overlapping of shapes or gaps.

**Rule 2:** The tiles must be regular polygons.

Did your group find any of these? If so how many?

Naming Tessellations

In your groups discuss how you would name some of the tessellations you found. Will your group name them based on what shapes they are made from, how many shapes they use, or another method?

Mathematicians only name regular and semiregular tessellations. First we count the number of sides that the shape we are using has. In the first case we have a hexagon, a six sided polygon. Then we count how many of this shape are around each point where the angles come together. We can name this tiling by listing the number of sides each shape has i.e. $(6, 6, 6)$ for the first tessellation. It is best to pick a point and write the number of sides each shape has in the center of the shape. It is then easy to go back and write the numbers in order.
We can use this notation to also name tilings with more than one shape. We start with the shape with the fewest number of sides. We then go around the point where the angles come together in a clockwise manner and label the other shapes in order. Try naming some of the tessellations you found.

It is important that the tessellation be regular of semiregular since each vertex must be the same in order to have an accurate name. If you attempt to name a tessellation which is not of this type you may get two names for the same pattern since each vertex is not the same, which is incorrect.
3.1.4 Lesson Three: Tessellations and Symmetry

Learning Objectives:

1. Find translation and reflection symmetries in tessellations.

2. Discover use of interior angles in determining which tessellations will form a regular or semiregular tessellation.

In this lesson we will look at what are called “regular tilings.” They are called regular tiling because they use regular shapes. For a shape to be “regular” it must follow two rules, all of its sides must look the same and all of its angles must look the same.

Necessary Definitions:

Rotation: The act of turning an object around a center point.

Translation: Moving an object in a specific direction with no rotation or distortion of the shape. What might be described as a slide along direction of motion.

Reflection: In a physical sense reflections are represented by lines that when the image is folded along them will create mirror images. These lines are called lines of symmetry.

Pass out the sheet from the Appendix titled “Tessellation printout to accompany Lessons 3”. Ask the students to fold along the red line. (You could also use a Myra for this to demonstrate reflection). Hold the paper up to the light and notice how the shape lines up with itself. This is called the “line of symmetry” because the symmetry occurs across a line. These figures are said to have “reflective symmetry” because the figure can be reflected about this line. Ask the students if they can find
any other lines of symmetry in the shapes? Have them draw them in with their pencil. (If you are using an overhead projector you may want to have the students come up and draw the lines on the transparency.)

Now have the students try to put the shapes together, use only one type of shape at a time. (Use the shapes in the Appendix. You can either cut some of them out beforehand (recommended) or have the students cut them out themselves. Each student or group of students should have at least one or two of each shape.) Make sure that the edges line up and the angles meet. Trace some of these shapes. If they repeat this process they can make what are known as regular tilings. Now take these tilings and have them try to find some lines of symmetry. Fold along these lines to make sure that they work. See how many lines the students can find in each tiling. Try to have them find at least two in different directions.

Now using just the hexagons and triangles try to fit the shapes together so that you find a pattern that will repeat in all directions. This is what is called a semi-regular tiling. See if they can find any lines of symmetry in this tiling. (There are two possible regular tilings that they can find with these shapes, only one has reflective symmetry.) Now give the students only the squares and triangles and see what tessellations can be formed. (There are also two regular tilings with these shapes, both of which have reflective symmetry.)

Now see if the students can find any rotational symmetries. Start with using the triangle tessellation (3, 3, 3, 3, 3) Have the students draw another copy of their tessellations on a second sheet of paper or a transparency. Have the students place
the first drawing on top of the second so that they line up. Then have the students place their pencil on a point of intersection (where vertices come together) on the top drawing. This point is called the center point of rotation, call this point A. Then rotate the transparency about this point until it “fits back on itself?” Have them try a different point of intersection to see if the same thing will happen. Now see if the students can find any points on the sides of or inside the triangle that will allow the tessellation to do the same. Ask them to find as many of these points as possible. Repeat this procedure with some of the other tessellations.

Lastly, have the students look at translational symmetry. Often time this is overlooked as a symmetry because we focus on one object rather than a repeating pattern. If we focus on our point A, we can see that there are many directions to “slide” or translate the transparency to have the pattern “fit on itself?” See how many the students can find for this tessellation. Repeat the procedure with other tessellations using more than one type of shape to elaborate on the topic.

3.1.5 Lesson Four: Styrofoam Stamp Tilings

**Learning Objective:** Create unique stamps which produce tessellations of irregular shapes.

1. Begin the lesson by refreshing the students memory of the basic shapes, triangle, square, rectangle, etc.

2. Give each student, or group of students, a square or rectangular piece of Styrofoam.
3. Have the students cut a small portion off the right side (you may want to trace the line before cutting) and carefully attach it to the left side with masking tape. See the figure below for guidance.

4. Repeat this process with the bottom and attach it to the top. See the figure below for guidance.

Option A

5. Now for the creative part. Have the students look at the new shape and decide what it reminds them of. Begin by tracing with a pencil then push a little harder to make indents. This will create a stamp that the student can use to create a tiling.

6. Have the students color the stamp with the colors they wish the tiling to have. Try to make the top a different color than the bottom, right different from left.
so that they will not blend together when the tiling is made. Another option is to paint the stamp all one color but to change the color with each placement of the stamp.

7. Use the stamp in the center of the page then repeat and place a stamp above and below, to the left and right of the center stamp.

8. Once this is finished they can touch up the stamped figured with more paint or add detail with markers, crayons, colored pencils, etc.

*Option B (Less Messy)*

5. Using our Styrofoam figure as a template trace the figure in the center of a sheet of paper.
6. Trace the figure again above and below, the left and right of the original figure.

7. Now for the creative part. Look at the shape created and decide what it looks like to you. Use a pencil to fill in details on your traced figure. You can use markers, crayons, paint, etc. to add color and more detail to your drawing.

Conclude the lesson by asking the students if they think the pattern will cover the whole page with this repeating pattern. Ask also what other figures they can see around them that cover a whole area in a repeating pattern.

3.1.6 Lesson Five: Play-Doh Stamp Tilings

**Learning Objective:** Create unique stamps which produce tessellations of irregular shapes.
1. Begin the lesson by refreshing the students' memory of the basic shapes, triangle, square, rectangle, etc.

2. Give each student, or group of students, some Play-Doh. On a sheet of wax paper or plastic wrap ask them to make a thin sheet from the Doh, about $\frac{1}{2}$ inches thick, using a rolling pin, a book, or their hands.

3. Now ask them to cut out either a square or a rectangle with a plastic knife.

4. Then have them cut a small portion of the right side and carefully attach it to the left side, as shown below.

5. Repeat this process with the bottom and attach it to the top. They can use some scrap Doh to help secure the edges.

6. The student will now have a stamp that can be used to create a tiling. Place a sheet of paper and a book on top of the doh and let this dry on the plastic
wrap over night. The book will help to keep the stamp flat and not dry out too much. (Do not let the shapes dry for more than two days!)

Option A (Messy)

7. After the stamp has dried slightly give the stamp a base coat of Tempera paint so that it will not absorb too much paint later.

8. Now paint the stamp with the appropriate colors. Try to make the top a different color than the bottom, right different from left so that they will not blend together when the tiling is made.

9. Use the stamp in the center of the page then repeat and place a stamp above and below, to the left and right of the center stamp. More stamps can be added as well.

10. Once this is finished you can touch up the stamped figured with more paint or add detail with markers, crayons, colored pencils, etc.

Option B (Less Messy)

7. Using the Doh figure as a template have the students trace the figure in the center of a sheet of paper.
8. Trace the figure again above and below, to the left and right of the original figure.

9. Now for the creative part. Look at the shape created by the Doh figure and ask the students to decide what it reminds them of. Use a pencil to fill in details on the traced figure. They can then use markers, crayons, paint, etc. to add color and more detail to the drawing.

Conclude the lesson by asking the students if they think the whole page can be covered with this repeating pattern. Ask also what other figures they can see around them that cover a whole area in a repeating pattern.

3.1.7 Lesson Six: Tessellations and Polyhedra

**Learning Objectives:**
1. Discover relationships between tessellations and polyhedra.

2. Examine relationships with regular polyhedron duals, truncation and semiregular polyhedra.

The previous lesson discussed regular tessellations of the plane. This lesson will discuss tessellations and their relationship to polyhedra. These are all activities that should be done in groups of 3 - 4 students.

Recall that in order for a shape to tessellate the plane the sum of the vertex angles must be 360°. With this in mind answer the following questions.

How many equilateral triangles surround each vertex? 

How many squares surround each vertex?

How many regular hexagons surround each vertex?

*Necessary Definitions:*

**Polyhedron** - In geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges. The word derives from the Greek poly (many) plus the Indo-European hedron (seat). The plural of polyhedron is ”polyhedra” (or sometimes ”polyhedrons”). A polyhedron is said to be regular and convex if its faces and vertex figures are regular polygons.

First let’s look at how to make some shapes that might tessellate space, namely, regular polyhedra. One key factor when deciding if a regular polygon will form a regular polyhedron is its interior angle and how many of the shape can be placed around a vertex. A good question to ask yourself is, “what is the minimum
number of polygons needed around a vertex in order for them to fold up and create a polyhedra vertex?” Look at the three regular tessellations to see if we can get some clues as to which polygons will form polyhedra when “folded up.”

The next question is, what is the upper limit criterion for a shape to create a polyhedron? It has been discussed that in order for a shape to tessellate the plane the sum of the vertex angles must be 360°. What limit must be put on the sum of the vertex angles in order for a polygon to create a regular convex polyhedron?

Which polygon(s) might form regular polyhedra? ________________

Which polygon(s) will definitely not form regular polyhedra? ___________

Are there any other regular polygons that might form regular polyhedra? _______________

With the polygons given begin by using the ones that you think will fit together to form a regular polygon. Tape together the shapes to make each regular polyhedra. The only rules are that:

1. Only one shape can be used when creating each polyhedra

2. All vertex angles must be congruent

3. That there can be no gaps or overlaps with the polygons.

With this completed now use the remaining shapes to create various prisms, pyramids and other polyhedra which use more than one type of regular polygon and hence are not considered regular polyhedron, but semiregular.
Lesson Seven: Tessellations of the Third Dimension

**Learning Objective:** Determine three-dimensional tessellations.

The first lesson discussed regular tessellations of the plane. The second lesson discussed how tessellations are related to polyhedra. This lesson will discuss tessellations in the third dimension. While the first lesson discussed why tessellations are important to cover a surface, this lesson will explore how some structures tessellate three-dimensional space.

Begin by looking around the room and see if you can identify any three-dimensional tessellations. List some of them: ____________________________

1. Using the figures created in the previous lesson discuss with your group which of the five regular convex polyhedra, the platonic solids, can be stacked together to fill the room without and gaps or overlapping of polyhedra. Start by using only one type of figure then explore what happened when you use more than one type of figure.

How many did your group find? ____________________________

List the polyhedra that work wither by then self or with another and how many are needed to surround a vertex. ____________________________

2. Now using the rest of the shapes previously created including prisms, pyramids, and other polyhedra see if any other tessellations of three-dimensional space can be created. Again start with just one type of figure (i.e. triangular prism, etc) then work with multiple types of figures. See if any relation can be made be-
tween the tessellations of the plane and tessellations of the third-dimension.

How many did your group find? _______________________________________

List the polyhedra that work wither by then self or with another and how many
are needed to surround a vertex. _________________________________

Was there any relation between the tessellations of the plane and tessellations
of the third-dimension?

3.1.9 Lesson Eight: Tessellations and Symmetry II

**Learning Objective:** Find translation, rotation and reflection symmetries in tes-
sellations.

**Necessary Definitions:**

**Rotation:** The turning of an object or coordinate system by an angle about a fixed
point.

**Translation:** A transformation consisting of a constant offset with no rotation or dis-
tortion. What might be described as a slide along an axis of motion.

**Reflection:** The operation of exchanging all points of a mathematical object about a
line of symmetry.

**Glide Reflection:** A product of a reflection in a line and translation along the same
line. Example: footprints
Look at some of the tessellations created in the first lesson (or print out sheet from Appendix titled “Tessellation printout to accompany Lessons 8”). Try to find lines of symmetry and centers of rotational symmetry. Lay the transparency of the tessellation so it lines up exactly with the sketch on the paper. Begin by looking at the regular triangular tessellation (3, 3, 3, 3, 3, 3) and answering the following questions about this tessellation:

**Translation:** If we focus on point A, is there any direction to slide or translate the transparency to have the pattern “fit on itself?”

If we consider the directions up and down to be one axis of motion, how many axes can I move the pattern along in order to have it “fit back on itself?”

**Rotation:** Now place your pencil on point A on the transparency and rotate the transparency until it “fits back on itself?”

Are there any points on or inside the triangle that will allow the tessellation to do the same, if so how many?

What is the largest number of times, n, that we can rotate the transparency around one of these points before we get back to the original position? We can use this number to find the smallest angle of rotation by taking $360^\circ/n$.

**Reflection:** Does the tessellation have any lines of symmetry, lines that when the image is folded along them will create mirror images?

How many of these lines can you find?

**Glide Reflection:** Does the tessellation have any glide reflections?
Look at the other tessellations on the page. Answer the questions above for each of these and fill in the appropriate data for the table below.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Number of Directions to Translate</th>
<th>Smallest Angle of Rotation</th>
<th>Number of Reflection</th>
<th>Glide Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.3.3.3</td>
<td>6</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>4.4.4.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.6.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.6.3.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.4.6.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.8.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.3.4.3.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.12.12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.3.3.4.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.12.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.3.3.3.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do all of the tessellations have translational symmetry? 

Is there any relationship between the polygons used in each tessellation and the smallest angle of rotation?
3.1.10 Lesson Nine: Tessellations and Wallpaper Groups

Learning Objective: Relate symmetry in tessellations to that of wallpaper patterns.

Answer the questions in the flow chart found in the Appendix titled “Flow Chart to Determine Wallpaper Patterns” based on the information we filled in on the table from Lesson 4 to determine which group each tessellation belongs to. Note that some may belong to the same group.

For example the first tessellation listed \((3, 3, 3, 3, 3)\)’s smallest rotation is \(60^\circ\) and has a reflection so it belongs to the \(p6m\) group. Try filling in the rest of the table.

<table>
<thead>
<tr>
<th>Tessellation</th>
<th>Wallpaper Pattern</th>
<th>Tessellation</th>
<th>Wallpaper Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3, 3, 3, 3, 3))</td>
<td>(p6m)</td>
<td>((4, 4, 4, 4))</td>
<td></td>
</tr>
<tr>
<td>((6.6.6))</td>
<td></td>
<td>((3.6.3.6))</td>
<td></td>
</tr>
<tr>
<td>((3.4.6.4))</td>
<td></td>
<td>((4.8.8))</td>
<td></td>
</tr>
<tr>
<td>((3.3.4.3.4))</td>
<td></td>
<td>((3.12.12))</td>
<td></td>
</tr>
<tr>
<td>((3.3.3.4.4))</td>
<td></td>
<td>((4.12.6))</td>
<td></td>
</tr>
<tr>
<td>((3.3.3.3.6))</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tessellations can represent different wallpaper groups if colored in specific ways. For instance the semiregular tessellation \((3, 3, 3, 4, 4)\) has two directions for reflections and a smallest angle of rotation of \(90^\circ\) yielding a wallpaper group pattern.
of \textit{cmm} (Figure 3.6(a)) while the coloring shown in Figure 3.6(b) is the wallpaper group \textit{p1}. Other examples of this are (3, 3, 3, 3, 3), (6, 6, 6) (Figure 3.6 (c,d)) and (3, 12, 12) which are all \textit{p6m} if uncolored but \textit{p3m1} if colored in a specific manner. Also, the semiregular tessellation (3, 3, 3, 6) which is usually \textit{p6} becomes \textit{p3} when colored as in Figure 3.6(f)

Figure 3.6: Semiregular tessellation create different wallpaper groups depending on coloring

(3, 3, 3, 4, 4) top (a) \textit{cmm} (b) \textit{p1}

(3, 12, 12) middle (c) \textit{p6m} (d) \textit{p3m1}
3.1.11 Lesson Ten: Construct Regular Polygon Tessellations Using Cabri II

1 Learning Objectives:

1. Create regular polygon tessellations by rotation, reflection and translation.

2. Determine which regular polygons tessellate.

3. Determine what conditions allow a polygon to tessellate.

Instructions:

1. Create an equilateral (regular) triangle. [Use regular polygon tool.]

   What is the interior angle measure? ____________

2. Rotate the triangle around one vertex by the interior angle measure. ² Continue this process until you are able to determine whether this polygon tessellates the plane. Remember: there should be no gaps or overlapping. [Use Numerical Edit, and Rotate tools.]

---

¹Lessons 10, 11 and 12 were adapted from labs designed by Dr. Antonio Quesada for the Concepts of Geometry Course at The University of Akron

²To create a regular triangle, select the Regular Polygon tool. Click on the screen to create the center point. Rotate your mouse clockwise until a regular triangle appears, then click. Select the Numerical Edit tool from the display toolbox. Type in the interior angle measure of the regular polygon. Press CRTL + U and select the degrees unit. Select the Rotate tool from the transform toolbox. Click on the polygon, click on the angle measure, then click on one of the vertices of the polygon. Repeat this process on the newly constructed polygon.

   To rotate the polygon 180° about a midpoint, select the Numerical Edit tool from the display toolbox. Type in the value 180. Press CTRL + U to see a list of units. Select degrees. Select the Rotate tool from the transform toolbox. Click on the polygon, click on the numerical value, then click on the midpoint of any side. Repeat the process by finding the midpoint of one of the new sides created and rotating about that point.
3. How many rotations did you make with the equilateral triangle in order to reach your conclusion? ___________ Explain how you could guess numerically this number without performing the rotations? ____________________________

4. Repeat the directions in #1, #2 & #3 for the regular shapes below

<table>
<thead>
<tr>
<th>Name of Polygon</th>
<th>Number of Sides</th>
<th>Vertex Angle</th>
<th>Does it Tessellate?</th>
<th>Number of Rotations Needed?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral Triangle</td>
<td>3</td>
<td>120°</td>
<td>Yes</td>
<td>6</td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Pentagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Hexagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Heptagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Octagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Nonagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Decagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Hendecagon</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regular Dodecagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Which of the regular polygons above tessellate?
6. Based on the information collected in the table above, what conclusion can you draw about when a regular polygon will tessellate? Express your answer mathematically.

3.1.12 Lesson Eleven: Construct Semiregular Polygon Tessellations with Multiple Polygons Using Cabri Geometry II.

Learning Objectives:

1. Create semiregular polygon tessellations by rotation, reflection and translation.

2. Determine which regular polygons form semiregular tessellations.

3. Determine what conditions allow polygons to tessellate together.

Definition: For a Semiregular tessellation the same rules must be followed from the first lesson. No two of the tiles have any interior point in common (no overlapping of shapes), no gaps, and the collection of tiles must completely cover the plane. We now have the additional rule that each vertex must look the same.

1. Open the file Regular Polygons. This file has all the regular polygons from triangles to dodecahedra that you will need to create semiregular tessellations.

2. Design some of your own tessellations using the given shapes and following the three rules above. Choose any given shapes and place them around a chosen vertex. Begin by reflecting some of the figures about an edge. See how
many tessellations are created with this method. [Use copy, paste, Rotate and Reflection tools.]

3. When you feel that you have exhausted this method choose different shapes to rotate around a given vertex. What is the necessary condition for this method to work? [Use copy, paste, Rotate and Reflection tools.]

4. You can try to use as many different shapes as you like to make each tessellation. See how many different tessellations you can make, but try to make at least 4 different tessellations. How many tessellations did your group create?

5. Once you have made a few semiregular tessellations make sure that they are really semiregular. Compare some of the vertices to see if they all look the same. Discuss with another group what you’ve found. Did you both find the same kinds of tessellations? Did your group find some that the others did not?

6. Which of the regular polygons from the table will tessellate together?

7. Based on the information collected in the table above and through Lesson 2, what conclusion can you draw about when a regular polygon will tessellate together? Express your answer mathematically.

3.1.13 Lesson Twelve: Construct Polygonal Tessellations Using Cabri II.

Learning Objective: Create nonregular polygon tessellations by rotation and translation and determine what conditions allow a polygon to tessellate.

1. Draw a scalene triangle.[Use Triangle tool]
a. Find the midpoint of each side. [Use Midpoint tool.]

b. Rotate the triangle about any midpoint 180°. Continue this process until the screen is filled or the polygons overlap. [Use Midpoint, Numerical Edit, and Rotate tools.]

c. Does the triangle tessellate? Explain your conclusions using your knowledge of the sum of the interior angles of a triangle.

d. Will all triangles tessellate? __________

2. Draw a convex or concave quadrilateral.

a. Follow the same process described in #1.

b. Does the quadrilateral tessellate? Explain your conclusions by using your knowledge of the sum of the interior angles of a quadrilateral.

c. Will all quadrilaterals tessellate? __________

3. It is possible for some pentagons to tessellate. Repeat the previous process for a pentagon. If the pentagon you create does not tessellate, change the shape using the hand tool until the pentagon will tessellate. When creating the pentagon, label the vertices to distinguish the original pentagon. Why do some pentagons tessellate when others do not?
3.2 Assessments

3.2.1 Assessment One: Naming Tessellations and Finding Symmetries

Name: ___________

Draw at least 2 lines of symmetry, which are not going in the same direction, in each of the tilings below. Then name them using the number method described Lesson Four.

3.2.2 Assessment Two: Determine All Semiregular Tessellations

1. In Lesson 10 we explored the use of multiple different polygons to tessellate the plane. In this activity we made sure that each vertex looked the same. Using Cabri and the table from Lesson 9, determine all possible tessellations that follow this rule.
2. What happens if we remove the restriction of same vertices? How do you think that this will affect your results from part (a)?

3.2.3 Assessment Three: Microsoft Paint Activity

In Lesson 11 we explored what happens when we use polygons to tessellate the plane. Now let's see what we get when we modify a regular polygon to form a new shape. Use Microsoft Paint to complete the following.

1. First, choose a color and draw a box about 1 inch in the center of your screen and fill the box with the same color used to draw the box.

2. Use Free-Form Select then click the bottom-most button shown below to make sure that your shape lines up correctly. Then on the left side of the box you've just created, draw an irregular shaped line from the upper left corner of the box, letting it end between half-way down the box and the bottom. The drawn line will disappear and a dotted box will remain.

3. Click and drag this shape to the opposite side (if you cut from the left side, move to the right side). Carefully align shape so there is no space in between and the tops corners line up.
4. Use Free-Form Select and inside the box on the bottom, draw an irregular shaped line from right corner of the box, letting it end about halfway across the bottom side. Release the mouse button. Again the drawn line will disappear and a dotted box will remain.

5. Click and drag this shape to the top of the box, carefully aligning it with the corner.

6. Now look at shapes to see what you’ve created. Try to visualize what the shape could be, a flower, plant, animal, etc. Lines, dots, patterns, backgrounds, etc. can be added later if you like. Be creative!
7. Use Free-Form Select to draw a circle around the entire shape. Copy this shape. Change the color of the new shape at this point, using the Fill With Color tool. Now Paste the figure previously copied in the window. It should appear in the top left corner of the screen and be the same color as the original figure.

8. Click and drag the shape to the location where it will fit on the top of the original shape and carefully align before releasing the mouse button. You can now change the color of the second shape if you wish.

9. Repeat Step 8 and fit to the other side of original shape. Continue until you have four shapes around the center shape. You may, however, choose to add MORE than just the four shapes.
10. At this point, you should have five or more tiles. Notice that if we were to repeat the pattern we could cover the whole page and create a tessellation.

3.2.4 Assessment Four: Journal Activity

1. List all definitions and properties that you have learned in this activity.

2. List any application of the tessellations that you can think of.

3. List any examples of tessellations that you might find in nature.

4. List the regular polygons that tessellate, either by themselves or with another polygon shape.

5. Can you relate this topic/concepts with other(s) previously studied? Explain your answer.

3.3 Teacher Resources and Selected Answer Keys

The following worksheet should be filled out after discussion of polygons, central and vertex angles. It is helpful to complete the Lessons on tessellations.

We’ve discussed many things about polygons and angles. Use the following formulas to fill in the chart below and determine the necessary angles for each polygon. These formulas should follow from their definitions. Note: These formulas can only be used for regular polygons. The equilateral triangle is filled in as an example.

*n* – the number of sides a polygon has

Central Angle = $360^\circ/n$
Vertex Angle = $\frac{180(n-2)}{n}$

Exterior Angle = $180^\circ$ – vertex angle

In other words Vertex Angle + Exterior Angle = $180^\circ$ since they are supplementary

<table>
<thead>
<tr>
<th>Name of Polygon</th>
<th>Number of Sides</th>
<th>Central Angle</th>
<th>Vertex Angle</th>
<th>Exterior Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral Triangle</td>
<td>3</td>
<td>$120^\circ$</td>
<td>$60^\circ$</td>
<td>$120^\circ$</td>
</tr>
<tr>
<td>Square</td>
<td>4</td>
<td>$90^\circ$</td>
<td>$90^\circ$</td>
<td>$90^\circ$</td>
</tr>
<tr>
<td>Regular Pentagon</td>
<td>5</td>
<td>$72^\circ$</td>
<td>$108^\circ$</td>
<td>$72^\circ$</td>
</tr>
<tr>
<td>Regular Hexagon</td>
<td>6</td>
<td>$60^\circ$</td>
<td>$120^\circ$</td>
<td>$60^\circ$</td>
</tr>
<tr>
<td>Regular Heptagon</td>
<td>7</td>
<td>$51.4^\circ$</td>
<td>$128.6^\circ$</td>
<td>$51.4^\circ$</td>
</tr>
<tr>
<td>Regular Octagon</td>
<td>8</td>
<td>$45^\circ$</td>
<td>$135^\circ$</td>
<td>$45^\circ$</td>
</tr>
<tr>
<td>Regular Nonagon</td>
<td>9</td>
<td>$40^\circ$</td>
<td>$140^\circ$</td>
<td>$40^\circ$</td>
</tr>
<tr>
<td>Regular Decagon</td>
<td>10</td>
<td>$36^\circ$</td>
<td>$144^\circ$</td>
<td>$36^\circ$</td>
</tr>
<tr>
<td>Regular Hendecagon</td>
<td>11</td>
<td>$32.7^\circ$</td>
<td>$147.3^\circ$</td>
<td>$32.7^\circ$</td>
</tr>
<tr>
<td>Regular Dodecagon</td>
<td>12</td>
<td>$30^\circ$</td>
<td>$150^\circ$</td>
<td>$30^\circ$</td>
</tr>
</tbody>
</table>

Do you notice any relationship between the Exterior Angle and the Central Angle of regular polygons? Yes, they have equal angle measure

Explain why this relationship occurs? The central angle divides a regular $n$-gon into $n$ isosceles triangles. Each triangle's apex has an interior angle of $a = \frac{360^\circ}{n}$
since it is the central angle. The base angle are congruent and equal to \((180^\circ - a)/2\),

but each interior angle \(i\) is really two of these base angles hence \(a\) and \(i\) are

supplementary. By definition the exterior angle is the supplemantray angle of the

interior (vertex) angle hence the central and exterior angles have equal measure.

3.3.1 Lesson Six: Tessellations and Polyhedra

The previous lesson discussed regular tessellations of the plane. This lesson will

discuss tessellations and there relation to polyhedra. These are all activities that

should be done in groups of 3 - 4 students.

Recall that in order for a shape to tessellate the plane the sum of the vertex

angles must be 360°. With this in mind answer the following questions.

How many equilateral triangles surround each vertex? \(6\).

How many squares surround each vertex? \(4\).

How many regular hexagons surround each vertex? \(3\).

Necessary Definitions:

Polyhedron - In geometry, a polyhedron is simply a three-dimensional solid which

consists of a collection of polygons, usually joined at their edges. The word derives

from the Greek poly (many) plus the Indo-European hedron (seat). The plural of

polyhedron is ”polyhedra” (or sometimes ”polyhedrons”). A polyhedron is said to be

regular if its faces and vertex figures are regular (not necessarily convex) polygons.

For this lesson we will only focus on the convex regular polyhedra.
First let’s look at how to make some shapes that might tessellate space, namely, regular polyhedra. One key factor when deciding if a regular polygon will form a regular polyhedron is its interior angle (use previous Polygon Handout) and how many of the shape can be placed around a vertex? A good question to ask is “what is the minimum number of polygons needed around a vertex in order for them to fold up and create a polyhedra vertex?” 3 (At this point the instructor will want to give some visual examples and explain why it is necessary for at least three polygons to surround a vertex.) Look at the three regular tessellations to see if we can get some clues as to which polygons will form polyhedra when “folded up.”

The next question is, what is the upper limit criterion for a shape to create a polyhedron? It has been discussed that in order for a shape to tessellate the plane the sum of the vertex angles must be 360°. What limit must be put on the sum of the vertex angles in order for a polygon to create a regular convex polyhedron? The sum of the interior angles must be less than 360°.

Which polygon(s) might form regular polyhedra? equilateral triangle, square

Which polygon(s) will definitely not form regular polyhedra? hexagon

Are there any other regular polygons that might form regular polyhedra? yes, pentagon

Now that it’s been established which shapes might form regular polyhedra give the students some of the shapes (not nets) to tape together to make the regular polyhedra. Give them the rules that only one shape can be used when creating each polyhedra, all vertex angles must be congruent and that there can be no gaps or
overlaps with the polygons. (They should find the three triangular polyhedra: tetrahedron, octahedron, and icosahedron, the square prism or cube and the dodecahedron which uses pentagons.) Give the students 10 - 20 minutes to complete this activity.

The proof can then be given of why there are only five convex regular polygons. Previously it has been discussed that in order for a group of polygons to form a regular polyhedra the sum of there angles must be less than $360^o$. We also know that if a polygon has $n$ number of sides that its vertex angle is $180^o(n - 2)/n$. So if there are $m$ polygons surrounding a vertex we have:

$$\frac{m(180^o(n - 2))}{n} < 360^o$$

$$m(n - 2) < 2n$$

$$mn - 2m - 2n < 0$$

$$mn - 2m - 2n + 4 < 4$$

$$(n - 2)(m - 2) < 4$$

We know that every polygon has at least 3 sides so $n > 2$. The only integers that satisfy the inequality above are as follows:

With this completed then have the students create various pyramids and other polyhedra which use more than one type of regular polygon and hence are not considered regular polyhedron but semiregular.

This should take another 10 - 15 minutes.
3.3.2 Lesson Eight: Tessellations and Symmetry

**Necessary Definitions:**

**Rotation:** The turning of an object or coordinate system by an angle about a fixed point.

**Translation:** A transformation consisting of a constant offset with no rotation or distortion. What might be described as a slide along an axis of motion.

**Reflection:** The operation of exchanging all points of a mathematical object about a line of symmetry.

**Glide Reflection:** A product of a reflection in a line and translation along the same line. Example: footprints

**Activity:**

Look at some of the tessellations created in the first lesson. Try to find lines of symmetry and centers of rotational symmetry. Lay the transparency of the tessellation so it lines up exactly with the sketch on paper. Begin by looking at the regular tri-
angular tessellation $(3, 3, 3, 3, 3)$ and answering the following questions about this tessellation:

**Translation:** If we focus on point A, is there any direction to slide or translate the transparency to have the pattern “fit on itself?”

If so, how many direction can I move the pattern in to have it “fit back on itself?”

**Rotation:** Now place your pencil on point A on the transparency and rotate the transparency until it “fits back on itself?”

Are there any points on or inside the triangle that will allow the tessellation to do the same, if so how many?

What is the largest number, $n$, of times we can rotate the transparency around one of these point before we get back to the original position? We can use this number to find the smallest angle of rotation by taking $360°/n$.

**Reflection:** Does the tessellation have any lines of symmetry, lines that when the image is folded along them will create mirror images? How many of these lines can you find?

Look at the other tessellations on the page. Answer the questions above for each of these and fill in the appropriate data for the table below.
<table>
<thead>
<tr>
<th>Translation</th>
<th>Number of Directions to Translate Point A</th>
<th>Smallest Angle of Rotation</th>
<th>Number of Reflection</th>
<th>Glide Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.3.3.3</td>
<td>6</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>4.4.4.4</td>
<td>4</td>
<td>90°</td>
<td>4</td>
<td>No</td>
</tr>
<tr>
<td>6.6.6</td>
<td>6</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>3.6.3.6</td>
<td>6</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>3.4.6.4</td>
<td>6</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>4.8.8</td>
<td>8</td>
<td>90°</td>
<td>4</td>
<td>No</td>
</tr>
<tr>
<td>3.3.4.3.4</td>
<td>4</td>
<td>90°</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>3.12.12</td>
<td>12</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>3.3.3.4.4</td>
<td>4</td>
<td>180°</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>4.12.6</td>
<td>12</td>
<td>60°</td>
<td>6</td>
<td>No</td>
</tr>
<tr>
<td>3.3.3.3.6</td>
<td>6</td>
<td>60°</td>
<td>None</td>
<td>No</td>
</tr>
</tbody>
</table>
3.3.3 Lesson Nine: Frieze and Wallpaper Groups

Answer the questions in the flow chart below based on the information we filled in on the table from Lesson 4 to determine which group each tessellation belongs to. Note that some may belong to the same group.

For example the first tessellation listed \((3,3,3,3,3,3)\)’s smallest rotation is \(60^\circ\) and has a reflection so it belongs to the \(p6m\) group. Try filling in the rest of the table.

<table>
<thead>
<tr>
<th>Tessellation</th>
<th>Wallpaper Pattern</th>
<th>Tessellation</th>
<th>Wallpaper Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3.3.3.3.3</td>
<td>p6m</td>
<td>4.4.4.4</td>
<td>p4m</td>
</tr>
<tr>
<td>6.6.6</td>
<td>p6m</td>
<td>3.6.3.6</td>
<td>p6m</td>
</tr>
<tr>
<td>3.4.6.4</td>
<td>p6m</td>
<td>4.8.8</td>
<td>p4m</td>
</tr>
<tr>
<td>3.3.4.3.4</td>
<td>p4g</td>
<td>3.12.12</td>
<td>p6m</td>
</tr>
<tr>
<td>3.3.3.4.4</td>
<td>cmm</td>
<td>4.12.6</td>
<td>p6m</td>
</tr>
<tr>
<td>3.3.3.3.6</td>
<td>p6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3.3.4 Assessment One: Naming Tessellations and Finding Symmetries

Figure 3.7: Solutions to Assessment One
CHAPTER IV
TESSELLATION LESSONS FOR KINDERGARTEN THROUGH COLLEGE

The following is a description of how the lessons found in Chapter 3 can be used at each grade level. Many of them can be used across various age groups while others are limited to certain grades due to content. The recommendations for approach of lesson have been tested to improve the lessons only and not to determine effectiveness of the lessons. These lessons were developed and adapted to allow students a creative and inquiry-based approach to learning certain topics in geometry, symmetry and algebra.

4.1 Kindergarten Through Fourth Grade

Students in this age group are beginning to learn and become comfortable with many new concepts. Interactive and creative lessons are often desired to keep students attention. Lessons one through four are best suited for younger children. Lesson five, though similar to four requires more maturity in the group of students to be completed correctly and affectively.
According to the Ohio Academic Content Standards[15] and NCTM standards [6], students should reach the following benchmarks by the end of second grade:

1. Describe, create and identify plane figures.

2. Identify, explain and model (superposition, copying) the concept of shapes and figures being congruent.

3. Describe location, using directional (above, below, right, left) and positional (first, last) words.

4. Identify and draw figures with line symmetry.

Lessons one and two can be used to explain the first two benchmarks. Lesson two is also very good for reiterating the names of plane figures and the number of sides and angles each has. Lesson four will help students develop the concepts of directional and positional descriptions in a creative manner, while lesson three deals with symmetry as well as superposition. It is important to make sure not to introduce concepts which may be too advanced for younger students so translation and reflection should be the only symmetry discussed in lesson three with rotation omitted. Learning objectives for this set of lessons and assessments:

1. Create regular polygon tessellations using hands on manipulatives.

2. Learn naming scheme for regular and semiregular tessellations.

3. Create unique stamps which produce tessellations of irregular shapes.
4. Find translation and reflection symmetries in tessellations.

Students in grades three and four are expected to know the previous benchmarks as well as

1. Describe, identify and model reflections, rotations and translations using physical materials.

2. Describe motion or series of transformations that show two figures are congruent.

So the rotation portions of lesson three can now be used at this grade level.

4.2 Fifth Through Eighth Grade

Middle grade students should be able to identify and label angle parts so lessons involving interior angles, such as lesson one in conjunction with the polygon reference sheet, can now be incorporated. The Ohio Academic Content Standards[15] say that students should reach the following benchmarks by the end of the eighth grade:

1. Identify and label angle parts and the regions defined within the plane where the angle resides.

2. Identify, describe and classify types of line pairs, angles, two-dimensional figures and three-dimensional objects using their properties.

3. Predict and describe results (size, position, orientation) of transformations of two-dimensional figures.
The first six lessons are ideal for teaching students these benchmarks as well as the following learning objectives:

1. Create regular polygon tessellations using hands on manipulatives.

2. Learn naming scheme for regular and semiregular tessellations.

3. Create unique stamps which produce tessellations of irregular shapes.

4. Discover use of interior angles in determining which tessellations will form a regular or semiregular tessellation.

5. Discover relationships between tessellations and polyhedral.

These lessons can be taught in order of appearance. Teachers may choose lesson four for students in fifth grade, while the Play-Doh lesson can be used for more mature classes for additional creativity with a more malleable material. Lesson six would work best for seventh and eighth graders who have become more familiar with polyhedra and other three-dimensional figures.

4.3 High School

Students in high school can be broken up into two groups, those taking a required geometry class and those in an honors section or integrated math class. The Ohio Academic Content Standards[15] and NCTM standards [6] state that the following benchmarks should be reached by all students at the end of the tenth grade:
1. Formally define geometric figures.

2. Draw and construct representations of two- and three-dimensional geometric objects using a variety of tools, such as straightedge, compass and technology.

3. Represent and model transformations in a coordinate plane and describe the results.

4. Prove or disprove conjectures and solve problems involving two- and three-dimensional objects represented within a coordinate system.

5. Establish the validity of conjectures about geometric objects, their properties and relationships by counter-example, inductive and deductive reasoning, and critiquing arguments made by others.

Lessons one, two, six and eight will work well for any group of high school students in teaching the learning objectives:

1. Create regular polygon tessellations using hands on manipulatives.

2. Categorize different tessellations into regular, semiregular or neither.

3. Learn naming scheme for regular and semiregular tessellations.

4. Discover relationships between tessellations and polyhedra.

5. Examine relationships with regular polyhedron duals, truncation and semiregular polyhedra.
6. Find translation, rotation and reflection symmetries in tessellations.

Students should be placed in groups of two or three for the activities. Lessons should be given in the numerical order on consecutive days to make sure the concepts follow in logical order.

Students who have demonstrated a clear grasp of this first set of lessons can then be presented with lessons seven and nine. Learning objectives for this set of lessons and assessments:

1. Determine three-dimensional tessellations.

2. Relate symmetry in tessellations to that of wallpaper patterns.

If teachers choose to use these additional lessons they should be taught in the order of lesson one, two, six, seven, eight then nine. Lesson seven follows well after lessons one, two and six since the students have just discussed two-dimensional tessellations and their relationship to polyhedra. Lesson nine draws heavily from the symmetry in each tessellation to determine which wallpaper group it belongs to, hence it works well after lesson eight.

4.4 College

The college lessons are split into two subsections. The first discusses the lessons appropriate for undergraduate students who are pursuing a degree in elementary education or related field in education of young children. These lessons will be more
in depth than those used for young students but less intense than those presented to those pursuing college degrees in math or secondary education. The second subsection discusses lessons suitable for college students pursuing a math degree or secondary education in mathematics certification.

4.4.1 Elementary Education Majors

The purpose of these lessons is to give students who wish to become elementary teachers not only a possible lesson that they might give when they begin teaching but more than enough information to go above and beyond what they might be required to teach. Lessons one - three along with the polygon reference sheet are relevant for students at this level.

Learning objectives for this set of lessons and assessments:

1. Create regular polygon tessellations using hands on manipulatives.

2. Categorize different types of tessellations.

3. Learn naming scheme for regular and semiregular tessellations.

4. Determine which polygons form regular tessellations.

5. Determine methods to create tessellations using irregular shapes.

6. Find translation, rotation and reflection symmetries in tessellations.

These lessons should follow instruction or review of regular polygons including discussion or interior (vertex) and exterior angles. For lesson one students can be grouped or left to work on the lesson alone to create tessellations and then grouped to
discuss results. Lesson two follows from the discussion at the end of lesson one and works best if a few examples are given and the students asked to name the remaining tessellations. Lesson three is effective when the instructor begins by completing a few on an overhead projector, then allows the students time to work on the remaining examples.

4.4.2 Math or Secondary Education Majors

There are two sets of lessons which can be used at the college level for math and math education majors. The first set, lessons five - nine, are some of the same used in the high school section. These lessons may necessitate some extra background information. These lessons would be ideally suited for an integrated math course but would also work well for an introductory geometry course especially if accompanied by a course in abstract algebra.

Learning objectives for this set of lessons, activities and assessments are:

1. Create regular polygon tessellations using hands on manipulatives.

2. Categorize different tessellations into regular, semiregular or neither.

3. Learn naming scheme for regular and semiregular tessellations.

4. Find translation, rotation and reflection symmetries in tessellations.

5. Relate symmetry in tessellations to that of wallpaper patterns.

The second set, lessons ten - twelve, is completely geometric and could be used for a college geometry class. Using lessons ten - twelve, students will create
polygon tessellations by rotation using geometry software. Students will work with both regular and irregular polygons in these lessons. A computer lab or set of calculators equipped with Cabri Geometry II is needed for each student or group of students to complete the lessons as well as a starter file which contains the properly sized polygons in addition to the worksheets provided in Chapter Two.

Students should have prior knowledge of the following key words: regular polygons, rotation, reflection and tessellations. If students have no experience with tessellations then the first part of lesson one (without the activity) can be given to introduce the topic. Students should also have existing knowledge of regular polygons, interior angles in a polygon, and the midpoint of a segment.

Learning objectives for this set of lessons are:

1. Create regular polygon tessellations by rotation, reflection and translation.

2. Determine which regular polygons tessellate.

3. Create nonregular polygon tessellations by rotation and translation.

4. Determine what conditions allow a polygon to tessellate.

The suggested procedure for this set of lessons is to put students in groups of three. Review the definitions of convex and nonconvex polygonal regions as well as introduce tessellations to students if necessary. Then have students complete the lesson/labs to discover the properties of various types of tessellations.

The second set, lessons five - nine, are some of the same used in the high school section. These lessons do not require a computer lab or calculators but may
necessitate some extra background information. These lessons would be ideally suited for an integrated math course but would also work well for an introductory geometry course especially if accompanied by a course in algebra.

Learning objectives for this set of lessons, activities and assessments are:

1. Create regular polygon tessellations using hands on manipulatives.

2. Categorize different tessellations into regular, semiregular or neither.

3. Learn naming scheme for regular and semiregular tessellations.

4. Find translation, rotation and reflection symmetries in tessellations.

5. Relate symmetry in tessellations to that of wallpaper patterns.

A discussion of some of the results observed when presenting these lessons is described in the following chapter.
CHAPTER V
RESULTS AND CONCLUSIONS

The lessons designed for this paper were presented to students in third and eighth grades, high school as well as two different college classes. The first college class was for pre-service elementary teachers while the second was senior level geometry class for math and math education majors. The information gathered from these presentations was used for improvement of the lessons only. The reader should be cautioned that the testing of lesson effectiveness has been left as future research.

This chapter will discuss some of the results gathered during the presentations. This includes examples of work completed by students at each grade level to give an idea of what can be expected from presenting the lessons. Many of the results are quite impressive and creative.

5.1 Third Grade

Lessons one, three and five were presented to one class of approximately 30 students over three days. Lesson five was started on day one and completed on day two as explained in further detail in the lesson. On day one the students created their Play-Do tiles and left them to dry on the window ledge of the classroom. Unfortunately, it was winter and the shapes froze rather than drying and did not work as well as the
original test. It was also noticed that the students at this age were too concerned with making the shape resemble a specific shape and hence adjusted the original form too drastically to fit back together. This result lead to the development of the Styrofoam lesson for younger students. The few tiles that did work were used by the teacher to demonstrate the process of creating the tilings to the entire class.

Lesson five only required half of the first day’s period so during the remaining time lesson one was started. Basic definitions were given to the class and a discussion of how polygons create patterns concluded day one. The students were very interested in this topic and excited to learn new words such as dodecagon. On the final day students were asked to find some tessellations of their own. They were only given the triangle, square and hexagon. They began by finding tiling with only one type of shape (the regular tessellations) and then given the task of finding some that use more than one type of polygon. With these shapes and tilings the students were then introduced to the concept of reflective symmetry first in a shape and then in the tessellations they had just created. Once the students seemed to be comfortable with each new step they were asked to come to the board and draw in the lines of symmetry that they could find. This made the lesson more interactive than originally planned.

5.2 Eighth Grade

Lessons one, three and four were presented to an honors class of 23 students in one day. The students had previously covered the topic of tessellations so lessons one
and three were discussed quickly to give some background information behind the purpose of lesson four. The students were very excited about using art in a math class. The students were each given a 2 inch x 4 inch piece of Styrofoam. The Styrofoam used was a lid from an egg carton and was approximately 1/8 inch thick, ideal for making the stamps. Students were very creative with their stamps making their stamps resemble anything from bathtubs to a man’s face and seen in Figures A.5. Overall, I was very impressed with the students creativity and careful work. Some students were not able to completely cover the entire page in the allotted time, choosing to return during study hall to finish their work.

![Stamp and Stamped Tessellations](image)

Figure 5.1: A stamp and a stamped tessellations made by eighth graders

5.3 High School Geometry

Lessons one, two, six, seven and eight were presented to two high school classes of 30 students each. The students were between the ages of 14 and 17 in an honors geometry class. I presented the lessons to the first period class with their teacher.
present. The teacher then presented the same lessons to the second class. Students were given a brief review of polygons and angles prior to the first day of lessons and asked to complete the polygon reference sheet.

On day one the students were presented with lesson one which gives the definition of a tessellation, the rules it must follow and a brief history. The students were then broken into groups of three to five with the task of creating tessellations with the shapes found in the Appendix. The students were given 30 minutes to find as many tessellations as possible. The first class found all of the regular and all but one of the semiregular tessellations. They overlooked the ability for a group of shapes to be able to create more than one tessellation and hence did not find $3.3.4.3.4$. The second class however, was able to find all three regular and all eight semiregular tessellations.

The second day the class began by discussing the classifications that they had come up with using lesson two, many of which depended on the number of shapes used in the tiling, and then were given the standard mathematical classifications. After this the students were lead through a series of inquiry-based questions to determine a relationship between a polygons ability to tessellate and its interior angle. The remaining time was given to lesson six. The students were very creative with this lesson and based many of their polyhedra off of the tessellations they had created the previous day. We were very impressed with the determination of some students to complete very involved polyhedra such as the great rhombicubactahedron and the truncated dodecahedron and seen in Figures 5.2(a) and (b). They continued
building their solids on day three and between the two classes were able to create all five regular polyhedra (the Platonic solids) and eight of the 14 semiregular as well as various prisms and pyramids.

![Examples of semiregular polyhedra created by high school students](image1.jpg)

**Figure 5.2:** Examples of semiregular polyhedra created by high school students
(a) Great Rhombicubicubactahedron (b) Truncated Icosahedron

![High school students create the truncated platonic solids](image2.jpg)

**Figure 5.3:** High school students create the truncated platonic solids

During the last half of the final day students were gathered into one group
to discuss tessellations of the third dimension and the duals of the platonic solids (Figures 5.3). The students were very interested to find out that many of the semiregular solids they had found could be formed by truncating some of the simpler regular polyhedra, and how truncated pieces could be merged together to form another solid. (Figure 5.3).

![Example of tessellating polyhedra, the truncated cube and tetrahedron.](image)

Figure 5.4: Example of tessellating polyhedra, the truncated cube and tetrahedron.

To conclude the lesson students were asked to complete assessments two and three, which yielded some very creative and colorful irregular shape tessellations. Examples of assessment results as well as more polyhedra can be found in the Appendix.

Overall, I feel that this set of lessons went particularly well. The students really enjoyed themselves while learning and the teacher asked for the resources to present the lessons to future classes.
5.4 College

5.4.1 Elementary Education Majors

Lessons one and two were presented to college students in the University of Akron’s Math for Elementary Teachers II course. A total of four classes and 100 students ranging from 18 to 45 completed the lessons and assessment three in the one hour and fifteen minute class period. The students were given a thorough background of polygons and angles before completing the polygon reference sheet.

Students were presented with lesson one including the definition of a tessellation, the rules it must follow and a brief history. The students were then broken into groups of four or five to complete the lesson with the shapes found in the Appendix. The students were given 50 minutes to find as many tessellations as possible. In all four classes students were able to find all three regular tessellations but usually only half of the semiregular ones as well as a common demiregular tessellation. The students were also asked to create their own classification which usually depended on the number of types of shapes. They were then introduced to the common classifications. Students quickly grasped the concept that in order for a polygon to form a regular tessellation its interior angle must divide 360°. They struggled however, with extending this concept to determine if a group of shapes would possibly tessellate based on whether the sum of the interior angles equals 360°.

The lesson was concluded with assessment three which produced some impressive results such as that shown in Figure 5.5. Students also mentioned that
they really enjoyed the lesson and that they would integrate Assessment Five in their future classrooms if possible.

Figure 5.5: Example of Assessment 3 results.

5.4.2 College Geometry

Lesson ten, eleven and twelve were presented to two classes, each of approximately 15 college juniors and seniors. Students were expected to have a previous knowledge of polygons and angles. The classes were held in a computer lab equipped with Cabri II. Students were given the lesson material via the internet and given 1$\frac{1}{2}$ hours to complete it in groups of two or three. The students were hesitant at first but once the relationship between a tessellating polygon and its interior angle were discovered the students progressed quickly, even more so with the irregular and nonconvex polygons. Most students completed the lesson in the designated time, the rest completed it for homework. Assessments Two and Four, along with the Journal Activity, were also completed for homework.
5.5 Conclusion

The purpose of presenting the lessons to various classes was mainly to improve the quality of learning for the students but not to determine effectiveness of the lessons. The were some technicalities in the wording of questions and instructions, which needed correction. The amended copies with these corrections are what is presented in this paper.

The descriptions listed above are to give the reader some examples of what can possibly be achieved when presenting the lessons from Chapter 3 to their own classroom. A fact that should be noted is that many of the of the classes that participated in the lessons and activities were honors classes. With that said, students in the classes mentioned were well prepared for the content material. Students in the classes really seemed to enjoy the activities and many of the teachers expressed that they would like to use the lessons again for future classes.
BIBLIOGRAPHY


Figure A.1: Shapes printout to accompany Lesson 1
Figure A.3: Tessellation printout to accompany Lesson 8
Figure A.4: Flow Chart to Determine Wallpaper Patterns [2]
A.2 Lesson Results

Figure A.5: A stamp and a stamped tessellations made by eighth graders

A.6(a)  A.6(b)

Figure A.6: Examples of polyhedra created by high school students

(a) Various prisms (b) Truncated Tetrahedron