PROBABILISTIC FINITE ELEMENT MODELING OF AEROSPACE ENGINE COMPONENTS INCORPORATING TIME-DEPENDENT INELASTIC PROPERTIES FOR CERAMIC MATRIX COMPOSITE (CMC) MATERIALS

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PROBABILISTIC FINITE ELEMENT MODELING OF AEROSPACE ENGINE COMPONENTS INCORPORATING TIME-DEPENDENT INELASTIC PROPERTIES FOR CERAMIC MATRIX COMPOSITE (CMC) MATERIALS

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ABSTRACT

The research included in this abstract pertains to probabilistic finite-element creep analysis of a composite combustor liner. A composite combustor liner is an aerospace engine component that is subjected to very high temperatures, ranging between 1500 - 2100 degrees Fahrenheit. A creep analysis of this component is essential for rational design as creep (a slow time-dependent information under constant load) is prevalent at high temperatures.

In a probabilistic analysis, many, if not all, of the state variables are represented by random variables with appropriate probability distributions incorporating relevant parameters. This formalism is much more realistic, as it more accurately describes the variability in properties and loadings that are inherent in the composition of aerospace materials and loadings encountered by aerospace components.
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“Reducing aircraft emissions is key to the U.S. aircraft industry’s remaining competitive in the global market. The Glenn Research Center, NASA’s lead center for aeropropulsion, is developing quieter, more fuel-efficient engines with fewer harmful emissions. NASA Glenn is studying new materials that can withstand the higher temperatures needed to reduce CO$_2$ emissions by 8 to 15 percent for large aircraft. These new-high temperature materials allow engines to run hotter and cleaner without increasing weight. Because of local concerns about the gases exhausted by airplanes, the expansion plans of several U.S. airports ‘Atlanta, Boston, Chicago, Houston, Los Angeles, New York, Philadelphia, Phoenix, and Washington’ have been stopped. Internationally, NO$_x$ emission limits set by the ICAO [The International Civil Aviation Organization] could result in strategies to limit air traffic, such as per flight emissions fees and curfews. Environmental concerns together with the prediction that world air traffic will grow 5 percent annually for the next decade have made reducing aircraft NO$_x$ emissions a priority.” [1].
1.1 Thesis objective

Within the overall NASA design framework, the question addressed by this paper is: how to determine safe life usage and general reliability of engine components built from new high temperature resistant materials? A second but equally viable question is: how to reduce the burden of costly component level destructive testing? Answering these questions involves taking material properties that are best described at a micro or lower level and applying them to models on a much larger, continuum scaled model. This is exemplified by the fact that new and innovative high temperature rig testing equipment had to be designed to determine the material properties for the various (Ceramic Matrix Composites) CMC's under development at NASA Glenn, [2].

This research presents a numerical model of aerospace engine components under complex loading conditions, [3, 4]. Aerospace components and structures have inherent randomness in their material properties since they are built from composite materials. For example, one might have a network of fibers which comprise the skeleton of the component. Surrounding these fibers is a bulk material, which may also be a composite material. The fibers are excellent at resisting pushing, pulling and twisting forces. However, the fibers may be susceptible to permanent changes of material strength when subjected to high temperatures. In metals, these changes are rarely beneficial to the overall strength of the material. Conversely, a bulk material, while contributing minuscule resistance toward mechanical loads, is highly resilient to large thermal loadings. The primary focus of this research is conducting a prob-
abilistic creep analysis on a given composite material using a finite element code. A comparative analysis of the strengths and weaknesses of various composites materials is outside the scope of this current work.

The numerical procedure presented is built from preexisting, widely accepted industry analysis methodologies, [5, 6, 7]. This permits for rapid and cost effective adaptation of these methods in a private industrial setting. There is a large class of applications that this numerical procedure can handle. Most notable are problems concerning alternate energy production and military hardware. Nuclear power plants have historically been the impetus for studying creep in metals. Since CMC materials offer high thermal resistivity, they are ideal for use in generating nuclear power. In addition, CMCs are light weight, which makes them attractive for use in wind turbines. On the military side, CMCs have the potential for use in lightweight armor and for heat resistant gun barrels, etc.

1.2 Overview of thesis

The second chapter contains a brief description of the theoretical background for this research. The chapter begins with a discussion of mechanics. Topics include both basic and more advanced material models. The next section explains the engineering concept of creep. This includes the basic theory, types of testing, characteristic of creep, and the Boltzmann Superposition Theory, [8, 9]. The difference between uniaxial and multiaxial creep models is then discussed. The third section in this chapter presents some information from probability and statistics. Wherein, probability
distribution functions (P.D.F)s and their related cumulative distribution functions (C.D.F)s are defined. In addition, a few commonly encountered distributions are presented. The section also contains a statement of the Central Limit Theorem and Bayes’ Theorem. The fourth section is meant to illustrate the connection between the use of deterministic models and probabilistic models. The last section pertains to reliability theory. The fundamental notations are presented, along with classifications of solution techniques. Some advanced topics are addressed; dealing with non-linearities in a system and correlation between a model’s variables.

The third chapter begins the process of constructing and integrating engineering models for the physical system studied. The first section presents the assembly of a gas turbine engine. While not directly related to the research, the section provides a backdrop for the implicit constraints that a designer must work within. The second section provides an introduction to the material, a ceramic matrix composite, used in this research. The third section introduces the model for the creep characteristics of the CMC incorporated in the model of the combustor liner. In addition, a test experiment designed to validate the numerical version of creep equations is presented. The fourth section contains the finite element model of the combustor liner. This includes a spatial discretization, loading and boundary conditions, and the variables which are treated probabilistically. The fifth section outlines the numerical methodology developed by NASA and used in this research, [3]. The last section is a survey of the reliability solution techniques that are encapsulated in the NESSUS code.

The fourth chapter begins with an analysis of the creep equations in this
research, applied to the simplified test brick model. Followed by a section with the results of the deterministic running of the FEA model. The last section of this chapter begins with the probabilistic assumptions and presents the results of the reliability analysis. The sections were split in this manner to emphasize the importance of the deterministic model for the probabilistic analysis. One benefits greatly by spending adequate time evolving the model at its mean values.

The last chapter reiterates the goal of this research and presents the conclusions obtained. The second section discusses the author’s personal contribution to the work. Then the limitations of this research are stated. Lastly, the possible extensions of this work are summarized.
CHAPTER II

THEORETICAL BACKGROUND

2.1 Mechanics

Stress, \( \sigma \), is the measure of a force, \( P \), acting on the material divided by, \( A \), the area of the material which is resisting that force. i.e. \( \sigma = \frac{P}{A} \). A deformation, \( \delta \), is the change in length of a material given loading conditions. Strain, \( \varepsilon \), is defined as the change in the deformation of a material divided by the length of the material. i.e. \( \varepsilon = \frac{\delta}{L} \), [10].

Figure 2.1A: Figure 2.1B:

Figure 2.1: Examples of deformation and stress, [10]
The above definitions are only valid in a one dimensional space. When designing in a three dimensional space, both stress and strain are further redefined into nine components:

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\] (2.1)

The first subscript denotes the plane for the cross-sectional area, \( A \), that is resisting the force. The second subscript denotes the axial direction in which the force is moving. So the \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}) \) are called the normal stresses (strains). Whereas the \( \sigma_{xy} = \sigma_{yx} \) \( (\varepsilon_{xy} = \varepsilon_{yx}) \), \( \sigma_{zx} = \sigma_{xz} \) \( (\varepsilon_{zx} = \varepsilon_{xz}) \), and \( \sigma_{zy} = \sigma_{yz} \) \( (\varepsilon_{zy} = \varepsilon_{yz}) \) are called shearing stresses (strains).

Consider a rod that is instantaneously loaded with a force (figure 2.1). If upon removing the force the rod returns to its original size, then the material is...
said to have behaved elastically for those loading conditions. Conversely, if upon
unloading of the force, the deformation is permanent then the material has behaved
plastically for those given loading conditions. It’s important to note that in these
examples, time has not been considered. Which is done to maintain a simple and
therefore mathematically tractable problem. For the elastic case, stress and strain
are related by the concept known as Hooke’s Law.

\[
\varepsilon_{xx} = \frac{1}{E} \left( \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \right) \quad (2.2)
\]
\[
\varepsilon_{yy} = \frac{1}{E} \left( \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) \right) \quad (2.3)
\]
\[
\varepsilon_{zz} = \frac{1}{E} \left( \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) \right) \quad (2.4)
\]
\[
\varepsilon_{xy} = \frac{1}{G} \sigma_{xy} \quad (2.5)
\]
\[
\varepsilon_{yz} = \frac{1}{G} \sigma_{yz} \quad (2.6)
\]
\[
\varepsilon_{zx} = \frac{1}{G} \sigma_{zx} \quad (2.7)
\]

Where \( E \) is Young modulus, \( \nu \) is Poisson’s ratio, and \( G \) is the shear modulus. \( E, \nu, \)
and \( G \) change per material and are generally determined through experimentation.
The set of these constants constitute a continuum scaled engineering model for a
component or structure. The version of Hooke’s law given above describes an isotropic
elastic material. Stress and strain are still related in plasticity and creep (to be defined
later) analysis, however not by such a simple relationship.

In the above discussion the material was assumed to be isotropic, which is
to say that the values of the stress and strain tensors are not direction dependent.
The following model, with slight modification from [12], is called orthotropic and was
used for material in this analysis:

\[
\varepsilon_{xx} = \frac{1}{E_{xx}} \sigma_{xx} - \frac{\nu_{yx}}{E_{yy}} \sigma_{yy} - \frac{\nu_{zx}}{E_{zz}} \sigma_{zz} \tag{2.8}
\]

\[
\varepsilon_{yy} = -\frac{\nu_{xy}}{E_{xx}} \sigma_{xx} + \frac{1}{E_{yy}} \sigma_{yy} - \frac{\nu_{zx}}{E_{zz}} \sigma_{zz} \tag{2.9}
\]

\[
\varepsilon_{zz} = -\frac{\nu_{xz}}{E_{xx}} \sigma_{xx} - \frac{\nu_{yz}}{E_{yy}} \sigma_{yy} + \frac{1}{E_{zz}} \sigma_{zz} \tag{2.10}
\]

\[
\varepsilon_{yz} = \frac{1}{G_{yz}} \sigma_{yz} \tag{2.11}
\]

\[
\varepsilon_{xz} = \frac{1}{G_{xz}} \sigma_{xz} \tag{2.12}
\]

\[
\varepsilon_{xy} = \frac{1}{G_{xy}} \sigma_{xy} \tag{2.13}
\]

The following relations also hold:

\[
E_{xx} \nu_{yx} = E_{yy} \nu_{xy} \tag{2.14}
\]

\[
E_{yy} \nu_{zy} = E_{zz} \nu_{yz} \tag{2.15}
\]

\[
E_{xx} \nu_{yx} = E_{yy} \nu_{xy} \tag{2.16}
\]

In an orthotropic model the material constants are further redefined to include information about three mutually orthogonal directions. For example, Young’s modulus, \(E\), is redefined as the quantities \(E_{xx}\), \(E_{yy}\) and \(E_{zz}\) for every point in the material. A simplified version of the orthotropic model is called transversely isotropic; where the material properties are isotropic in the x-y plane, but differ as one moves the along z axis.
2.2 Creep analysis

Creep is a time dependent inelastic deformation of a material. It is readily observable that high temperatures are influential for a material displaying this phenomenon, [8]. As temperature can be considered a measure of the amount of kinetic energy that exists at the molecular level, this is a measure of free energy in the mesh which facilitates the breaking of molecular bonds. Thus realistic simulations and design models of aerospace structures require a mechanism that incorporates creep, both for predicting safe lifetime use and for the manufacturing process.

For general analysis it is first assumed that creep strain, $\varepsilon^c = f(\sigma, t, T)$, is a function of time $t$, temperature $T$ and stress $\sigma$. Also, one normally assumes that the effects of time, temperature and stress are separable, which leads to a form of $\varepsilon^c = \sum_{i=1}^{n} f_i(t) g_i(T) h_i(\sigma)$, [8]. This allows the analyst to conduct a sensitivity analysis for the response of creep strain based on each variable. Predominately, materials’ creep data comes from one dimensional tension tests. This is due to the complexity and cost of performing time dependent high temperature experiments. Obviously, the objective is to match any given model’s predictions with the results from these one dimensional tensile creep tests. To do so, one employs analytical tools, as found in [13], or a numerical methodology, as detailed later in this text, to utilize this information.

It is important to be aware of the different types of testing available as the trends exhibited by the creep strain curve are highly dependent on the type of test per-
formed. While there exist many methodologies for collecting creep data, the methods can be sorted into two large classifications - constant load tests and constant stress tests. During a constant load test, as the specimen deforms, the stress levels increase since the material’s ability to resist the constant force is changing with the deformation. Conversely, in a constant stress test, the testing apparatus is forced to modify the force it is applying, in an effort to stay in equilibrium with the changes to the test specimen, [8].

![Diagram showing differences in types of creep testing](image)

Figure 2.3: Differences in the types of creep testing, [8]

Now the basic characteristics of creep will be discussed within the context of a one-dimensional isotropic model. These topics include the creep strain curve, strain hardening and stress relaxation. The theory will then be extended to multidimensional models where the distribution of creep strain no longer occurs uniformly across the material but may propagate in any direction within the material or component.
When analyzing a creep strain curve it is beneficial and common practice to break the curve into three regions. As there exist different phenomenological events in these regions which explain the expected curvature. A few caveats, the creep strain curve is highly dependent on loading profile for each specific material. Furthermore, it is quite possible that a given creep strain curve does not display all three ranges, (see figure 2.4 a, curve $C_3$). Lastly, it is not guaranteed that there exists a clear point of delineation between these regions.

![Figure 2.4 a, [8]](image1)

![Figure 2.4 b, [13]](image2)

Figure 2.4: Creep strain and creep recovery

In figure 2.4 a, the curve between points A and B displays expected trends within the primary or transient region. During this time period the material is experiencing the most rapid inelastic deformation with time. Also known as strain hardening, with these inelastic changes the component can maintain resistance to a load with less effort. Or said more formally, as the material strain hardens there is
a reduction in stress for a constant load. The other important observation is that even as the amount of accumulated creep strain is increasing, the rate at which the creep strain is accumulating is decreasing, [8]. This provides an explanation for the existence of strain hardening and is the impetus for calling this region the transient region. Some models attempt to capture information in this region, others do not.

The trends exhibited between the points B and C are referred to as steady state creep. Its important characteristic is that the creep strain rate has stabilized on its minimum value, [8]. In elasticity, every stress has an equivalent strain which it corresponds to, related through Hooke’s Law. One can view a similar phenomenon in a material that is creeping. The one-to-one correspondence still exists. However, the material’s ability to achieve equilibrium between stress and strain now requires some amount of time, whereas in elasticity such an adjustment by the material is considered to be instantaneous, [9]. Thus the strain hardening which characterized the primary region is the mechanism by which the material internally converges to an equilibrium between stress and strain for a given loading profile. Further, the material enters the steady state region once it has finished strain hardening. This region is most commonly the focus of mainstay models.

The tertiary region, point C to D, is conceptualized as the time period just prior to rupture of the material. Here one observes that the increases in stresses associated with changes in the surface area begin to overcome the beneficial effects of strain hardening, [8]. However, if constant stress testing is employed, then it is quite possible for the material to never enter this region, see figure 2.3 and 2.4 a. This
region is the least likely to be incorporated into mainstay models.

The models used to simulate creep can also be used as stress relaxation models. One considers the strain to be the independent variable and runs experiments to plot a stress history. This phenomenon was touched upon above. As a material strain hardens, there is a reduction or relaxation of stress in the material. Thus even as one conducts a stress relaxation analysis separately from a creep analysis, the underlying mechanism is the same, [8, 9].

The above discussion assumed a constant stress (load) and constant temperature. However, in design of practical structures one must plan for situations involving variable stresses and fluctuating temperatures. So it is important to capture the behavior of the material upon unloading and cycle loading situations. This leads to the topics of creep recovery and the Boltzmann superposition principle.

Creep recovery is the reduction or recovery of strain that occurs upon unloading of the component, see figure 2.4 b. The rationale underlying creep recovery was touched upon above. As a component is unloaded, the material needs some increment of time to re-establish a new state of equilibrium between stress and strain. First is the immediate release of the elastic contribution, $\varepsilon$, then the time dependent recovery, $\varepsilon^c$, occurs. Depending on the material there may be full recovery or only partial recovery of the creep strain.

The Boltzmann superposition principle is: “if a series of stresses are applied to a material at different times, each contributes to the deformation as if it alone were acting.”, [9]. This allows one to linearly sum the stresses, in time, which are acting
on the material to derive their combined strains. With slight modification from [9], one may write the total strain $\varepsilon(t)$ as:

$$
\varepsilon(t) = \sum_{i=1}^{n} \sigma_i \varepsilon^c(\sigma, t - t'_i, T)
$$

(2.17)

for different times: $t'_1, t'_2, \ldots t'_n$. The important consequence of this principle, is that inelastic materials have a memory of previous loading states. It is this memory that gives further insight to the concept of creep recovery. Lastly, the relevance of this principle to a finite element modeling endeavor will be presented later.

When extending an analysis from a one dimensional space into a three dimensional space, the contribution of stress from every direction needs to be incorporated into the creep equation. The creep model used in this study is a scalar equation, while stress and strain are rank two tensors, as depicted above (figure 2.2). Thus one defines an effective stress, $\sigma_e$, to resolve the mismatch. There is more than one way to define the effective stress. How one defines an effective stress is highly dependent on the type of material being analyzed. From [14], with slight modification to avoid confusion with notation used later, the following definition was used:

$$
\sigma_e = \sqrt{3J_2^*} + \sqrt{3\gamma \sigma_{kk}}
$$

(2.18)

$$
J_2^* = \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \psi[\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2]
$$

(2.19)

Where $\gamma \sigma_{kk}$ "incorporates the effects of hydrostatic stresses into the inelastic potential function." [14]. The constant $\psi$ is used to scale the shear stresses. Since the material used in this study is assumed to be orthotropic, "... the shear response is assumed to be uncoupled from and independent of the normal response." [14]. This study has
set $\gamma = 0$, since effects concerning hydrostatic stress are not considered. In [15], the following definition of effective stress can be found:

$$\sigma_e = \sqrt{2\Gamma_{ij}\Gamma_{ji}}$$ (2.20)

$$\Gamma_{ij} = s_{ij} - \frac{I}{2}(3D_{ij} - \delta_{ij})$$ (2.21)

$$I = D_{ij}s_{ji}$$ (2.22)

$$D_{ij} = d_i d_j$$ (2.23)

where $\delta_{ij}$ is the Kronecker delta, “$s_{ij}$ are the components of deviatoric stress, and $d_i$ ($i = 1, 2, 3$) are the components of a unit vector denoting the local fiber direction.”, [15]. As an alternate method, one may develop a creep model which has vectorial dimensions, [16].

2.3 Probability notations

A probability distribution function (P.D.F), $f_X()$, predicts the values a variable is likely to realize. A cumulative distribution function (C.D.F), $F_X()$, is related to a random variable’s (P.D.F), by:

$$f_X(x) = \frac{d}{dx} F_X(x)$$ (2.24)

$$F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt.$$ (2.25)

The (C.D.F) is the sum or integral of probabilities for all values less than or equal to some value of $x$, and has the following properties:
• A (C.D.F) is a monotone nondecreasing function with values between zero and one.

• The probability of negative infinity is zero.

• The probability of infinity is one.

• If \( x_1 < x_2 \), then \( F_X(x_1) < F_X(x_2) \)

A few commonly used (P.D.F)s from [17], are the normal, lognormal and weibull distributions, see 2.6 and 2.7. These curves have the following definitions:

The normal distribution:

\[
f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)
\]

where \(-\infty < x < \infty\) and \(0 < \sigma\).

The lognormal distribution:

\[
f(x; \mu, \sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & x < 0 \end{cases}
\]

Figure 2.5: A (P.D.F) and (C.D.F), [17]
The Weibull distribution is:

\[
    f(x; \alpha, \beta) = \begin{cases} 
    \frac{\alpha}{\beta} x^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^\alpha\right) & x \geq 0 \\
    0 & x < 0 
\end{cases} \tag{2.28}
\]

where \( \alpha, \beta > 0 \).

In the above distributions, \( \mu \), is the mean or expected value. It is a statistical measure of center for the interval about the random variable being analyzed. Its formal definition is

\[
    \mu_X = \int_{-\infty}^{\infty} x f(x) \, dx
\]

where \( f(x) \) is the (P.D.F) of \( X \). The variance \( \varsigma \) of \( X \) is

\[
    \varsigma_X^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \tag{2.30}
\]

Variance gives the analysts a statistical measure of the spread of values that a random variable may take. Most texts use the symbol \( \sigma \) to denote the variance of a random variable. \( \varsigma \) will be used since \( \sigma \) was defined above to denote stress in a material. Two
important theorems that will be invoked later are the Central Limit Theorem and Bayes’ Theorem.

Central Limit Theorem: “Let the function $Y$ be the sum of $n$ random variables $X_i$ ($i = 1, 2, \ldots, n$). Furthermore, assume the $X_i$ are statistically independent, and their probability distributions are arbitrary. The Central Limit theorem states that as $n$ approaches infinity, the sum of these independent random variables approaches a normal probability distribution, if none of the random variables tends to dominated the sum”, [18].

Bayes’ Theorem: “Let $A_1, A_2, \ldots, A_k$ be a collection of $k$ mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, \ldots, k$. Then for any
other event $B$ for which $P(B) > 0$, [17]

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)} \quad j = 1, \ldots, k'' \quad (2.31)$$

Where the $|$ denotes conditional probability. One reads $P(A|B)$ as the probability of an event $A$ occurring given that the event $B$ has already occurred.

2.4 Extending from deterministic methods to probabilistic methods

In a deterministic study, one first assumes that the material is comprised of one homogeneous material. Next it is assumed that the forces applied are uniform along the length of component. The strategy is to match the maximum intensity of possible forces to a material’s minimum ability to resist these forces. The goal is to find where the maximum deflection, vibration, etc. of the material occurs, and the magnitude of that response. Due to the inherent scatter of material properties, these basic assumptions are not applicable for composite materials. This leads to over engineering of the structure, since deterministic models cannot accurately describe the small scale material properties. This is exacerbated by the fact that composite materials are currently very costly to manufacture. Therefore it is necessary that these materials’ design models accurately predict their behavior.

To build a more accurate approximation for the randomness in the material composition and the complex nature of the loading conditions, solution techniques are extended from purely deterministic models to a probabilistic formalism. One begins by considering the relevant state variables as random variables instead of
taking on a specific value. A probabilistic discretization of the model is created by assigning a probability distribution function (P.D.F) to each random variable. On successive sample runnings of the model, one expects the values which are to be instantiated into the variables to differ. The strategy employed by probabilistic formalisms is to assume a (P.D.F) or employ data fitting techniques to approximate a (P.D.F) to each loading and material resistance variable. One is then concerned with constructing a (P.D.F) and (C.D.F) for the component or structure based on the (P.D.F)s’ of the basic variables. To arrive at this information one employs traditional deterministic solution techniques. Only the basic assumptions have been relaxed to permit uncertainty in the model’s design variables.

2.5 Reliability analysis

The objective of a reliability analysis is to predict the probability of failure $P_f$ of a structural component for given loading conditions, [19]. It is noteworthy that the following formalism can be readily applied to risk assessment analysis. The utility and particulars of many of the following definitions are dependent on the type of analysis employed. Also, one generally chooses an analytical method based on the amount of information known about the structural component and the complexity of the problem. So to help build the theory, consider the most simplistic reliability problem: let an uniaxial bar be loaded by only one stress, $S$, and let the bar have only one resistance, $R$, to that stress.
2.5.1 Fundamental definitions

Basic or primitive variables

In any modeling endeavor, one identifies a set of variables, for which changes in the response of the model are tracked relative to independent changes in these variables. For example, material properties, dimensions of the structure, and applied loads are all examples of basic variables, [19]. For the introductory problem, R and S are the basic variables. However, in general one formulates a vector, \( \mathbf{X} \), for which each component, \( X_i, \ i = 1, \ldots, n \), is a basic variable. Generally the basic variables will be random variables, but not all of them need be. In fact one might be obliged to conduct statistical testing in an effort to prove that any given variable can be treated as a deterministic variable. For these variables, the (P.D.F)s’ are either known, \( f_R() \) and \( f_S() \), or one assumes (initially a multi-normal distribution) the (P.D.F)s’ that the basic variables will follow.

Limit state equations

The “limit state equation”, is used to delineate the variable design space into safety and failure regions. Through this mechanism one encodes physical information about the structure into a quantifiable model. For the example problem the component has failed when, \( R - S < 0 \). In other words, the resistive strength, R, of the component is
less than the stress, $S$, it must endure. Thus the probability of failure becomes, [19]:

$$P_f = P(R \leq S)$$  \hspace{1cm} (2.32)

$$= P(R - S \leq 0)$$  \hspace{1cm} (2.33)

$$P_f = P\left(\frac{R}{S} \leq 1\right)$$  \hspace{1cm} (2.34)

$$= P(\ln R - \ln S \leq 1)$$  \hspace{1cm} (2.35)

A more general form can be expressed as,

$$P_f = P(G(R, S) \leq 0)$$  \hspace{1cm} (2.36)

Where $G$ is used to denote the “limit state function”. The most general form one might find in the literature has $n$ basic variables.

$$P_f = P(G(X) \leq 0)$$  \hspace{1cm} (2.37)

Most probable point

The most probable point (MPP or $x^*$), is a concept employed in many reliability methods. One begins with standard optimization techniques to find this point. Then the reliability analysis is performed in a neighborhood about this point in the basic variable space. This extra manipulation makes the reliability analysis computationally tractable. It is likely that a structure has more than one most probable point, [20]. However, for this analysis, it is assumed that the material has only one most probable point. Lastly, the most probable point is often referred to as the design point.
Figure 2.8: The most simplistic reliability problem, [18]

Reliability index

The reliability index, $\beta$, has different formulations based on the analytical method employed. A very famous version of $\beta$ was defined by Hasofer and Lind, [18].

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{s_R^2 + s_S^2}}$$ (2.38)

Where $R$ and $S$ are assumed to be uncorrelated and the multi-normal distribution, $\Phi$, has been imposed. The reliability index is the shortest distance from the origin of the normalized basic variable space to the limit state equation. If the above assumptions are true then the reliability index is the probability of failure.

$$P_f = \Phi(-\beta)$$ (2.39)

Conversely, if these assumptions do not hold, then the reliability index functions as an initial approximation and corrections for the probability of failure are sought.
2.5.2 Solution Techniques

To compute $P_f$, the probability of failure, one considers

$$P_f = P(G(X) \leq 0) = \int_{G(X) \leq 0} \cdots \int f_X(X) \, dX$$  \hspace{1cm} (2.40)

Where $f_X(X)$ is the joint (P.D.F.) for the vector of basic variables $X$. The objective is to integrate over the failure domain, $G(X) \leq 0$. Rarely can this integration be computed analytically. The methods to compute approximations to this integral can be classified into the following types: One can employ numerical integration schemes to approximate the multidimensional integral, [19]. As a different strategy one may impose a normal distribution onto each basic variable. Since tabulations for multi-normal distribution’s integrals are readily available. One can either assume that the variables follow a norm distribution or translate the known distribution into a normal distribution, [19]. In [21], a methodology is proposed which decomposes the domain of the non-normal basic variables into a set of sub domains that are small enough to be treated as normal distributions. The solution to the overall problem is then the sum of the integrals of the sub-domains.

2.5.3 Sensitivity analysis

When designing, it is important to understand how a system is influenced by its basic variables. This information is obtained through the reliability analysis. There is a slight difference from a deterministic sensitivity analysis. Here one receives the change in the probability of failure for the component relative to changes in the statistical
parameters of the design variables, i.e. from [22]:

\[
\frac{\partial P_f}{\partial \Theta_i}.
\]

(2.41)

2.5.4 Nonlinear limit state equations

As with most analysis, the complexity of methods employed are dependent upon the amount of non-linearity present in the system. Some of the methods are well suited to deal with non-linear limit state functions while others are not. One approach is to linearize the problem via a Taylor series expansion. Caution is required, as the subsequent analysis is dependent on choice of expansion point. One could use the mean, \( \mu_X \), of the design space, \( X \), as the expansion point. Also, the most probable point is often used as the expansion point. Recall that the most probable point is first found by using optimization techniques. With large amounts of nonlinear behavior in the system one expects multiple most probable points, \( \bar{x}^* \), [20]. Resulting in a need to conduct the analysis multiple times, one analysis per design point found in the system. Monte carlo and related methods handle non-linear limit states very well. As the problem is not solved analyticly, but approximated by a large number of trials for a given model. Another alternative is to employ optimization techniques. One might employ an analytic approach such as calculus of variations. Alternately the problem may be better suited by the use of a numerical schema, i.e. assuming a rough approximation of the limit state then refining the limit state equation with an iterative scheme.
2.5.5 Highly correlated basic variables

It is statistically important to test for correlation between the design variables. In a highly correlated design space, a systematic discovery of the model will not produce an accurate account of each variable. Given such correlations between the design variables one may use a Rosenblattt transformation, [19], to map the basic variable vector, $X$, onto $R$, where the components $r_i$ are no longer correlated. To construct $R$, Bayes theorem is systematically applied to each component $X_i$ of $X$. Once the Rosenblattt transformation has been conducted the reliability analysis can proceed as mentioned above. However, $R$ is a purely mathematical construction and has no physical link to the system studied. So once a (C.D.F) for the system has been constructed, it is best to derive the sensitivity factors analytically with respect to the original basic variables $X$. 
CHAPTER III

SPECIFICATION OF THE PROBLEM

3.1 Gas turbine engine components

Figure 3.1: A gas turbine engine assembly.[23]

"In the computer drawing, we have cut out a portion of the engine to have a look inside. Various parts on the photograph are labeled and the corresponding parts on the computer drawing are indicated. At the front of the engine, to the left, is the inlet which brings outside air into the engine. The F100 engine picture does
not show the aircraft inlet because the inlet is part of the airframe and changes from
aircraft to aircraft. At the exit of the inlet is the compressor, which is colored cyan.
The compressor is connected by a blue colored shaft to the turbine, which is colored
magenta. The compressor and the turbine are composed of many rows of small airfoil
shaped blades. Some rows are connected to the inner shaft and rotate at high speed,
while other rows remain stationary. The rows that spin are called rotors and the
fixed rows are called stators. The combination of the shaft, compressor and turbine
is called the turbo machinery. Between the compressor and the turbine flow path is
the combustion section or burner, which is colored red. This is where the fuel and
the air are mixed and burned. The hot exhaust then passes through the turbine and
out the nozzle. The nozzle performs two important tasks. The nozzle is shaped to
accelerate the hot exhaust gas. ...”, [23].

3.2 CMC materials

“NASA Glenn Research Center researchers have developed two high-performance
SiC/SiC composite systems with state-of-the-art thermostructural capability up to
1315 °C (2400 °F). These systems are based on advanced processes that significantly
improve the performance of commercially available SiC fibers, boron nitride (BN)
fiber coatings, and SiC matrices that are initially formed by conventional processes,
such as chemical vapor infiltration (CVI). ”, [24]. SiC/SiC stands for silicon-carbide-
fiber-reinforced silicon carbide matrix. The benefits of these materials include, high
temperature resistance, less corrosive in hostile environments, and lightweight. When
dealing with ceramics there are two classifications: monolithics and composite matrices. A composite matrix ceramic (CMC) is a monolithic ceramic plus reinforcing elements, see figure 3.2. The reinforcing agents can be classified as whiskers (short fibers), long continuous fibers, and particulates (chucks). Use of whiskers or particulates as the reinforcing agent leads to brittle behavior of the material. When using long continuous fibers, one is concerned with the degree in which the fibers are bonded to the matrix material. When the long continuous fibers have a strong bond with the matrix, the material will (given tensile loading) exhibit brittle behaviors. Conversely, when there exists slippage between the fibers and the matrix, the fibers can deform based on their properties, (instead of the properties of the overall composite). However, as the fibers deform, friction causes structured cracking throughout the matrix material. As these cracks progress, the matrix material no longer supports the load,
leaving the work solely to the fibers. Composite failure may result from failures in the matrix material or with the reinforcing elements. Much work is devoted to modeling, on a micro-level, the various damage modes present in these materials, [26, 27]. Since this research focuses on the continuum scale, statistical distributions are used to represent the scatter in the material properties at a macro-level. A given fiber has definite material properties, which influence and are influenced by the material properties of the matrix in local vicinity. This interrelationship of different micro-level material properties are aggregated to form marco-level (P.D.F)s of those material properties. This also lends credence to the use of a normal curve in describing certain material properties, as the area near the fiber will contribute the most with a tapering off of influence as one moves through the matrix material away from that fiber.

3.3 A creep model for CMC materials

The creep model, \( \varepsilon^c \), used in this research has a general form similar to

\[
\varepsilon^c = f(\sigma)(At + B[1 - \exp(-Ct)]) \tag{3.1}
\]

\[
\dot{\varepsilon}^c = f(\sigma)(A + BC \exp(-Ct)) \tag{3.2}
\]

The creep model contains transient effects and steady state effects. The decaying exponential models the transient effects, \( At \) models the steady state region, \( C \) has units of temperature. \( A \) and \( B \) are treated probabilistically and are specific to the particular CMC used in this sample analysis. The temperature loading conditions of the model partially determine the values of \( C \). The dot reflects differentiation with
respect to time, the rate form is used since creep strain is highly dependent on the path of the loading profile. The linear relationship between elastic stress and elastic strain is completely retained, the calculation of total strain is now the sum of the elastic strain and creep strain, \( \varepsilon(t) = \varepsilon^e(t) + \varepsilon_c \). The expected characteristics of creep recovery that the code needs to reproduce has been discussed in a prior chapter. The creep model and material properties of the CMC used in this analysis are courtesy of [24].

It is important to validate numerical version of the constitutive equations that describe creep prior to adding them to the general probabilistic finite element model. An analysis has been performed on a isotropic, three dimensional, eight node brick of unit volume. One solid element was used to represent an uniaxial testing coupon used in physical creep testing. An uniform but fluctuating (with respect to time) tensile pressure load was applied in the z direction. The brick was kept at a constant temperature of 1127 Kelvin. The four nodes, which comprise the face opposite of the pressure loading, were held fixed in the z direction. Each node has three degrees of freedom for displacements.

3.4 A finite element formulation of a combustor liner

Figure 3.3 B is a CMC combustor liner, figure 3.3 A is the spacial finite element discretization of the combustor liner, [3, 4]. Hot air and fuel enter on the left side of the liner and the combustion occurs in the middle of the liner. The following boundary conditions have been imposed: All nodes at the right most side of the
Figure 3.3: A CMC combustor liner and a finite element model, liner are held fixed relative to translations of the x direction (length of the liner). Furthermore, one node is unable to rotate around the x-axis. All other nodes are free to rotate and displace in all degrees of freedom. The fixed node is to avoid a rigid body motion, this condition is necessary in most finite element models. In regards to loading conditions: there is an uniform, outward directed, internal pressure load applied to all elements in the mesh. A thermal loading profile is applied along the length of the liner, see figure 3.4. The temperature profile and the uniform, outward directed, internal pressure load simulates the combustion. Think of the mesh as the collection of rings, attached left to right. The x-axis represents rings of elements along the length of the liner, the y-axis is the temperature at that ring of elements. There is a thermal gradient through the thickness of the liner, which expresses hotter temperatures inside the liner versus the temperatures experienced on the surface of
the outer liner. So each ring of the liner has an uniform temperature. The response variable tracked by this analysis is time to local creep failure of the component. Here failure is defined to be the time when creep strains reach a magnitude of .005 or greater. See table 3.1 for the probabilistic design variables used in this analysis.

The element chosen for this analysis “is a four-node, thick-shell element with global displacements and rotations as degrees of freedom. Bilinear interpolation is used for the coordinates, displacements and the rotations. The membrane strains are obtained from the displacement field; the curvatures from the rotation field. The transverse shear strains are calculated at the middle of the edges and interpolated to the integration points.”, [28]. Each node has six degrees of freedom, three degrees for rotations and three for displacements. The elements are orientated so that the elements’ rectangular face is in plane with the axial direction and normal to the thickness of the liner. The layers of the shell elements are used to differentiate between
Table 3.1: Design variable space

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient of thermal expansion</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>Thickness of the liner</td>
<td>$t$</td>
</tr>
<tr>
<td>Pressure load</td>
<td>$P$</td>
</tr>
<tr>
<td>Young’s modulus in the x and y direction</td>
<td>$E_{11} = E_{22}$</td>
</tr>
<tr>
<td>Young’s modulus in the z direction</td>
<td>$E_{33}$</td>
</tr>
<tr>
<td>Shear modulus in the x versus y direction</td>
<td>$G_{12}$</td>
</tr>
<tr>
<td>Shear modulus in the z versus y and z versus x directions</td>
<td>$G_{23} = G_{31}$</td>
</tr>
<tr>
<td>Thermal load along the length of the inner liner</td>
<td>$T_1$</td>
</tr>
<tr>
<td>Thermal load along the length of the outer liner</td>
<td>$T_2$</td>
</tr>
<tr>
<td>Creep constitutive equations constants</td>
<td>A,B,C</td>
</tr>
<tr>
<td>Scaling factor for shearing stresses</td>
<td>$\psi$</td>
</tr>
</tbody>
</table>

the inner liner and the outer liner. Eleven layers were used since this problem is highly non-linear: a result of both the temperature profile and the creep effects.

The material properties of the CMC, [24], are temperature dependent. However, the temperatures experienced in the model are outside the range of given data. Extrapolation was conducted using Newton’s interpolation polynomials for each material property. A divided differences tableau was used in the construction of these polynomials, [29]. The data given for the coefficient of thermal expansion had to be converted into an instantaneous coefficient of thermal expansion. For the given values
of $\bar{\alpha}_i$, a reference coefficient of thermal expansion $\bar{\alpha}_r$ was chosen. Then for all $i$ in the set,

$$\alpha_i = \bar{\alpha}_r + (\bar{\alpha}_r - \bar{\alpha}_i) \quad (3.3)$$

was used to convert the given data into an instantaneous coefficient of thermal expansion, [30].

3.5 The numerical methodology

The intent of this work is to present a methodology that effectively models continuum scaled structures with material properties that are best described at a micro or lower level. The “Probabilistic Structural Analysis Methods (PSAM) developed at NASA Glenn Research Center...”, [3], are ideal for modeling (SiC/SiC) matrix composite materials as there is inherent scatter in the material properties and loading conditions of these components. Furthermore, creep, the time-dependent inelastic response, is empirically verifiable in these materials at extreme temperatures. Thus our objective is to present the robust incorporation of these technologies in a feasible and re-useable fashion. In [31], two stochastic finite element methodologies are presented. This work has as a foundation the same structure as the second procedure outlined in that text. The major difference is the variable tracked in this analysis and the incorporation of the creep constitutive equations into the finite element model.

First the base model is run with each relevant basic variable taken at its mean value, instead of being scaled beyond a maximum or minimum value. Then the
model is used to create an implicit approximation of \( G(X) \), the limit state equation. This is accomplished by conducting a sensitivity analysis, where the parameters are chosen via probability distribution functions. With each running of the model, in the sensitivity analysis, one of the basic variables is systematically perturbed about its mean. For, \( i = 1, \ldots, n \), the \( i \)th variable is taken to the left of its mean by 5\% its standard deviation, \( \mu_{X_i} - 0.05 \cdot \sigma_{X_i} \). Then for the next run that variable is taken to the right of its mean, again by 5\% its standard deviation, \( \mu_{X_i} + 0.05 \cdot \sigma_{X_i} \). Each running of the model in the sensitivity analysis must instantiate values of the basic variables which have a realistic chance of occurring in the physical system.

In actuality, the percentage one spans to the left and right of the mean is the choice of the analyst. Five percent is an excellent starting point if very little is known about the distributions of the basic variables. As the analyst has more than likely begun the analysis by assuming a multi-normal form. As the standard normal curve is symmetrical about its mean (\( \mu = 0 \)), spanning by five percent the standard deviation captures a large range probable values. One loses the possible values found under the tails of the normal curve, which have increasingly diminishing probability of occurring. After each running of the probabilistic finite element model, one collects the values that were instantiated into the basic variables and the desired response. The datasets from the deterministic and probabilistic runs of the finite element model are then interpolated to form a response surface, [32], within the design variable space. Since the analysis includes both the left and the right sides of the mean, there is enough data to use quadratic interpolation rules.
3.6 Survey of reliability methods included in NESSUS

NESSUS (Numerical Evaluation of Stochastic Structures Under Stress) was used to build the response surface and to conduct the subsequent reliability analysis. One builds an input file for NESSUS, which includes the datasets built from the successive runnings of the mainstay finite element package, probabilistic definitions of each basic variable, and a requested reliability analysis type. The remainder of the chapter describes the reliability algorithms which are encoded in the NESSUS software. The following information comes from [22]:

3.6.1 First and second order methods

These analysis methods begin with the assumption that the (P.D.F)’s of the basic variables are normal curves. Further the limit state equation needs to be a linear function of the basic variables:

\[ G(X) = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n \]  \hspace{1cm} (3.4)

If the limit state equation is nonlinear, then one linearizes with a Taylor expansion centered about the most probable point. First order methods only use the linear terms:

\[ g_1(X) = a_0 + \sum_{i=1}^{n} a_i(X_i - X_i^*) \]  \hspace{1cm} (3.5)

Thus the reliability index is defined as:

\[ \beta = \frac{\mu_g}{\sigma_g} = \frac{a_0 + \sum_{i=1}^{n} a_i\mu_i}{\sqrt{\sum_{i=1}^{m} a_i^2\sigma_i^2}} \]  \hspace{1cm} (3.6)
Similarly the second order methods have a Taylor expansion of the following form:

$$g_2(X) = a_0 + \sum_{i=1}^{n} a_i(X_i - X_i^*) + \sum_{i=1}^{n} b_i(X_i - X_i^*)^2 + \sum_{i=1}^{n-1} \sum_{j=1}^{n} c_{ij}(X_i - X_i^*)(X_j - X_j^*)$$ (3.7)

FPI computes the second-order coefficients by numerical differentiation. These methods should not be used if the limit state equation is highly non-linear or implicitly defined.

3.6.2 Advanced first order methods

This method begins by determining the most probable point. Next a Taylor expansion is used to construct a quadratic function centered around the most probable point. The quadratic is then transformed into a linear function in a different variable space. The reliability analysis is conducted in the new variable space. The extra manipulation reduces the amount of information lost, due to truncation error of the Taylor expansion.

3.6.3 Fast convolution method

This method uses the same linearizing techniques as the advanced first order methods to create a linear limit state function. Then standard FFT routines are applied to conduct the probabilistic integration. For a given basic variable X,

$$H_X(\omega) = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x} \, dx$$ (3.8)

This method is theoretically better than the advanced first order method, however it is computationally more taxing. Since the NESSUS code applies the transform to all variables in the design space at once.

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3.6.4 Monte carlo method

Instead of integrating design variable space in the failure domain, monte carlo methods run many simulations of a given model and count the number of times the component failed. Let \( N \) be the total number of trials conducted, and \( n(G \leq 0) \) be the number of failures, then the probability of failure is

\[
p_f \approx \frac{n(G \leq 0)}{N}, \quad [19]
\]

(3.9)

This method works well with problems that contain highly non-linear characteristics, since the model is not “solved” explicitly. However this method is computationally intense since for each analysis large number of trials need to be preformed.

3.6.5 Radius based importance sampling methods

All importance sampling methods are refinements of the monte carlo method. One uses any additional information to reduce the amount of calculations needed. Thereby, reducing extra variance from the answers. This method uses a sphere or hypersphere to exclude points from sampling. The FPI code has two methods, ISAMF and ISAMR. In the ISAMF method, the user supplies a reduction factor \( F \). The most probable point is calculated and multiplied by this factor \( F \) to determine the radius of the safe region. In ISAMR, the user supplies a radius for the region to be excluded from the Monte Carlo analysis. ISAMR was included to give the analyst more control, as one can fall back to this method if FPI cannot find the proper most probable point.
3.6.6 Adaptive importance sampling methods

These methods iteratively adjust a limit-state surface which defines the sampling space. FPI has two methods, AIS1 and AIS2. FPI begins by locating the most probable point and converts the problem into the standard normal space. The reliability index along with the most probable point are used to create the initial sampling space. AIS1 bounds the sampling space with planes and AIS2 uses parabolic curves. A monte carlo analysis is incrementally implemented along with rules to adjust the limit-states. AIS1 uses translations of the bounding plane to widen the sampling space. AIS2 systematically modifies the curvatures of the parabolic surface to envelop the failure region.

3.6.7 Advanced mean value Methods

Mean value (MV) methods also use Taylor’s expansions to linearize the limit function. Here the expansion is taken about the system’s mean instead of the most probable point. An advanced mean value (AMV) analysis is a mean value analysis with the inclusion of a correction term. An advanced mean value plus (AMV+) analysis extends the advanced mean value method with an iterative scheme. Once the advanced mean value analysis has been conducted, a most probable point is constructed. The limit function is then linearized at the most probable point. The analysis continues with iterative refinements to the most probable point for the component.
CHAPTER IV

RESULTS

4.1 Deterministic uniaxial finite element analysis

This section describes an experiment designed to validate whether the encoded numerical version of the constitutive equations exhibits the proper creep characteristics. Figure 4.1 A shows the uniaxial creep behavior that this model produces. It was created by instantaneously loading the brick by a tensile load of 15 ksi. The brick was allowed to creep for twenty five hours, it was then unloaded and allowed to creep for another twenty five hours. Then another tensile load of 20 ksi was applied and the brick was allowed to creep for twenty five hours. Again the brick was unloaded and allowed to creep for an additional twenty five hours. Lastly, a tensile load of 5 ksi was applied and allowed to creep. One sees the transient creep stage, then upon unloading the elastic contribution is instantaneously released, however the current development does not incorporate the time dependent recovery expected from this inelastic material. Figure 4.1 B shows that the numerical creep model has the capacity for describing trends present in inelastic materials. This figure matches well with the results form the analytic model presented from [24]. It was produced by post processing the data of several runs of the unit brick model. Following the Boltzmann
Figure 4.1: Creep behaviors of the constitutive equation, Superposition Principle, three separate and independent runs of the model where made. The tensile loads were made at 15 ksi, 20 ksi, and 5 ksi. Another important assumption is that the compressive version of these loading profiles would produce data with the same magnitude but with the opposite sign. All three experiments were loaded into Matlab, and their compressive equivalents were created. The first twenty five hours of the plot is representative of the just the 15 ksi tensile load only. The interval of twenty five to fifty hours was created by allowing the 15 ksi tensile load to continue as before. However, the 15 ksi compressive load is now added to the tensile load starting at its time zero. At hour fifty the contributions of the tensile 20 ksi load begin to take effect. So, in the interval between fifty to seventy five hours, the 15 ksi tensile load produces values as if its simulation had run for fifty to seventy five hours. The 15 ksi compressive load is independently contributing to the curve its
values from the twenty five to fifty intervals. Lastly, the plot shows the creep strain data for 20 ksi tensile load in its zero to twenty five intervals, etc.

The difference in creep characteristics between figures 4.1 A and B illustrates a shortcoming in the current usage of the finite element software package. In [13, 30], there are auxiliary rules to be encoded into the finite element code. The finite element engine uses these auxiliary rules to recognize when stress reversals occur. Further, given a stress reversal, it maintains a moving origin, thereby giving the component a memory of previous states. Implementing a full creep analysis with stress reversals will be left as an extension of this work. For the full combustor liner, the temperature and mechanical loading profiles will not be fluctuating in time.

It is important that all the runs of the test model use the same time step, else the response cannot be meaningfully summed via the Boltzmann Superposition Principle. First, to determine the time step, the loading condition with the largest magnitude was run. All other independent runs of the test model were then forced to evolve with that time step.

4.2 Deterministic multiaxial finite element analysis

The deterministic running of the model about the mean values of the basic variables yields a time to local creep failure of around 2,000 hours. This occurs at the right side of the liner where the temperatures are the most elevated. However, the region of combustion also shows very interesting developments of creep strain corresponding to the hot spots that occur during the burn of the engine. Figures 4.2, 4.3, and 4.4
Figure 4.2: 100 hours of simulated time

Figure 4.3: 1000 hours of simulated time

displays the liner at approximately 100, 1000 and 2000 hours. The figures on the left (or A) panels shows the development of creep strain. The right (or B) panels shows
Figure 4.4: 2000 hours of simulated time

Figure 4.5: Creep strain at the failure node

a redistribution (or relaxation) of stresses in the regions where creep is occurring. In the region of combustion one sees creep strains begin in a manner that resembles point sources. Followed by an elliptical spreading of creep strain until a limiting
process sets in. This limiting process is explainable by the decreasing contribution of the transient stage of the creep model. Figure 4.5 shows creep strain vs. time at the node where the liner first experiences creep failure.

4.3 Probabilistic analysis of a combustor liner

One views each design variable as a random field instead of a determined value. The sensitivity analysis of finite element model is the embodiment of a probabilistic discretization of the problem. The choice of instantiated values for the design variables are determined from their known or assumed distributions and statistical parameters. With this information reliability methods determine the conditions prevalent in the most probable failure mode and the sensitivity of the system to each design variable.

It was assumed that every design variable has a normal distribution. Further it is assumed that all design variables are independent and uncorrelated. Clearly, this cannot be the case, as the creep model explicitly states strain as dependent to time, temperature, stress, and previous strain states. In addition, the given material properties are temperature dependent. The base finite element model was able to fully represent the temperature dependence of the material properties. However, the response surface which defines the limit state between failure and safety is built only in the region of the liner where the temperatures are the hottest. This simplification is justified since this region is the first to experience local failure. It is left as an extension to this work to decouple all correlations between the basic variables, using the statistical tools discussed previously.
Figure 4.6 A

Figure 4.6 B

Figure 4.6: Results from the reliability analysis,

A monte carlo analysis (figure 4.6 A) and an advanced first order analysis (figure 4.6 B) were preformed. The results of both are in agreement. Figure 4.7 has the sensitivity of the combustor liner. The most important factors are thickness of the liner, Young’s modulus in the x direction $E_{xx}$, the temperature load, the pressure load and the steady state creep constant. An increase in either the thickness or with Young’s modulus will increase the time to failure. Whereas, an increase in the other factors will lead to more rapid creep failure. The second echelon of factors include the scaling constants, $\psi$ for the shearing stresses, the primary creep constants and the Poisson’s ratios $\nu_{xy}$ and $\nu_{xz}$. It is interesting to note the disparately between the importance of the scaling constant $\psi$ and Poisson’s ratios $\nu_{xy}$ and $\nu_{xz}$. Of the least important factors are the Coefficient of Thermal Expansion and Young’s modulus in the z direction, $E_{zz}$. 

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Figure 4.7: The sensitivity factors
5.1 Summary

The objective of this research is the realistic numerical modeling of aerospace engine components built from temperature resistant CMC materials. This entails translating micro and smaller scaled material properties to macro level continuum models. Further, the elevated temperature loading conditions require incorporating creep characteristics of the material into the model. This study demonstrates that probabilistic finite element methodologies can be used in conjunction with creep (an advanced time-dependent mechanical phenomenon) analysis. First one solid element was used to model an uniaxial testing coupon. The results of this experiment matched well with the analytic model presented. However, upon unloading of the force (or a stress reversal), one does not observe creep recovery. Capturing this characteristic of an inelastic material has been left as an extension. The creep constitutive equations were then applied to the full combustor liner. One observes interesting developments of creep strain and corresponding stress redistribution in the region of combustion. The combustor liner was then probabilistically analyzed to determine the time to local creep failure. An approximation of the combustor liner was built; since the
physical problem is too complex to be represented by an analytic set of governing equations. Response surfaces are used to approximate the combustor liner, which is then used in a reliability analysis, to determine the probability of failure. For the example problem considered, one expects a 50% chance of local creep failure once the liner has been used for roughly two thousand hours. Further there is 99% chance of local creep failure after approximately two thousand six hundred hours of operation. The most important beneficial factors are the thickness of the combustor liner and Young’s modulus in the x direction, \( E_{11} \). The steady state creep constant, temperature and pressure loads are also relevant. Increases in this set of basic variables will lead to a decrease in the time to creep failure.

5.2 Personal contributions

To build the response surface used in the probabilistic portion, \( n \) basic variables requires \( 2n + 1 \) separate runs of the finite element model. This was accomplished via a mostly automated procedure using a set of control scripts. A mixture of Perl, Python, and Awk code was used manipulate all the input decks across multiple directories. There were pre-production setup scripts, run scripts and finally post-production data gathering scripts. A master file was maintained with a listing of all the variable names and their appropriate statistical parameters, mean, variance, etc. The master file is used by the controller scripts in all three stages of production. For a given analysis a subdirectory is selected to act as the home or “root”. This directory acts as a parent for a set of subdirectories, one for each probabilistic variable analyzed.
The setup scripts create this directory structure and place two modified versions of
deterministic input deck into each subdirectory. The template input deck is renamed
as the variable name and a prefix denoting whether the file’s namesake variable is
less than or greater than the mean. Since the input deck has a structured format,
it is a trivial exercise to script the use regular expressions to modify the appropriate
variable left or right of its mean value. The run script reads the master file, iterates
through the directory structure and calls the Marc finite element program for each
probabilistic run. The obvious benefit of this structure is that an error in one version
of the model does not cause the other versions of the model to stop running. Further,
a probabilistic variable can be added to the design space without difficulty, as one
needs only to add a subdirectory and run the two new cases. An obvious extension
would be to surround the control scripts with a user friendly interface.

5.3 Limitations

While the current model is valid for the current gas turbine engine, the finite element
formulation needs a mechanism to describe creep relaxation before it can simulate
the next generation of gas turbine engine. The current state of technology for gas
turbine engines is all on or all off. The next generation of gas turbine engine will
have multiple firing per second. In addition, while the finite element model was able
to handle various nonlinear behaviors, the reliability analysis was conducted with a
linearized model.
5.4 Extensions

- A creep analysis that properly accounts for stress reversals and cyclic loading conditions.


- Finding the correlation between the design variables.

- Analysis of the other engine components.

- Comparative analysis between differing CMC’s (Ceramic matrix composites).

- Refine the automation of the probabilistic analysis and provide a user friendly interface.

- Use of parallel computing algorithms.
BIBLIOGRAPHY


APPENDIX
NOMENCLATURE

Table A.1: Nomenclature

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{ij}$</td>
<td>Stress in tensor notation</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Deformation</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
<td>Strain in tensor notation</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Possion’s ratio</td>
</tr>
<tr>
<td>$G$</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>$\varepsilon^c$</td>
<td>Creep strain</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>Effective stress</td>
</tr>
<tr>
<td>$\dot{\varepsilon}^c$</td>
<td>Creep strain rate</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_{ij}$</td>
<td>Strain rate</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Scaling factor for shearing stresses</td>
</tr>
<tr>
<td>Variable</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Average or mean</td>
</tr>
<tr>
<td>$\varsigma$</td>
<td>Standard deviation</td>
</tr>
<tr>
<td>$X$</td>
<td>Random variables are capitalized</td>
</tr>
<tr>
<td>$x$</td>
<td>A realized value for a random variable</td>
</tr>
<tr>
<td>$\mu_X$</td>
<td>The mean of a random variable $X$</td>
</tr>
<tr>
<td>$\varsigma_X^2$</td>
<td>The standard deviation of a random variable $X$</td>
</tr>
<tr>
<td>$f_X$</td>
<td>P.D.F</td>
</tr>
<tr>
<td>$F_X$</td>
<td>C.D.F</td>
</tr>
<tr>
<td>$P_f$</td>
<td>The probability of failure</td>
</tr>
<tr>
<td>$G()$</td>
<td>A limit state equation</td>
</tr>
<tr>
<td>$x^*$</td>
<td>Most probable point</td>
</tr>
<tr>
<td>$\beta$</td>
<td>The reliability index</td>
</tr>
<tr>
<td>$\underline{x}$</td>
<td>The underline denotes a vector</td>
</tr>
<tr>
<td>$\Phi()$</td>
<td>The multi-normal distribution</td>
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