TEACHING CONCEPTS FOUNDATIONAL TO CALCULUS USING INQUIRY AND TECHNOLOGY

A Thesis

Presented to

The Graduate Faculty of The University of Akron

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

Benjamin David Marko

May, 2006
ABSTRACT

There is currently a push in the state of Ohio to enhance the teaching of concepts foundational to Calculus in grades 8-12. Evidence of this came in the spring of 2005 when the Ohio Department of Education called for grant proposals to develop programs that are focused on topics that are foundations to Calculus. Further, the National Council of Teachers of Mathematics has stressed the effective use of technology and inquiry in their past and current standards. This thesis contains inquiry-based lessons that explore concepts that would traditionally be taught in the Calculus classroom, but can be used in lower level courses such as Algebra. The use of the graphing calculator is essential for each one of these lessons to allow students the opportunity to explore more mathematical concepts in a less time consuming manner.
ACKNOWLEDGEMENTS

I would like to take this time to thank a number of people without whom this thesis would not have been possible. Thanks to my advisors, Dr. Antonio Quesada and Dr. Linda Saliga, for all of their guidance and effort in helping me complete this project. Thank you also to the entire Mathematics Department at the University of Akron for giving me the opportunity to pursue my degree while allowing me to teach classes. I am extremely grateful for everything the department has done for me over the past six years. I would also like to thank all of my friends and family. Especially, I thank my parents, Pam and Craig, for giving me their never-ending love and support over the years. Also, to my sisters, Jennifer and Natalie, and brother, Jonathan, I thank you for being great examples for me and showing me how important education is. Most of all, I thank my wonderful and beautiful wife, Katie. Your love, support, and encouragement have strengthened me so much over the past seven years. I am so grateful that God has given me a woman who will love and support me in all that I do. I am truly blessed to have you in my life.
# TABLE OF CONTENTS

## LIST OF FIGURES

| LIST OF FIGURES | vii |

## CHAPTER

### I. INTRODUCTION

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

### II. LITERATURE REVIEW

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Calculators</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Inquiry</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Functions, Variables, and Number Sense</td>
<td>10</td>
</tr>
</tbody>
</table>

### III. LESSONS, HOMEWORKS, AND ASSESSMENTS

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 The Properties of Functions</td>
<td>12</td>
</tr>
<tr>
<td>3.2 The Zeroes (Roots) of a Function</td>
<td>18</td>
</tr>
<tr>
<td>3.3 Solving Equations</td>
<td>23</td>
</tr>
<tr>
<td>3.4 Solving Inequalities</td>
<td>27</td>
</tr>
<tr>
<td>3.5 One-to-One and Onto Functions</td>
<td>30</td>
</tr>
<tr>
<td>3.6 Exploring the Basic Properties of Polynomial Functions</td>
<td>38</td>
</tr>
<tr>
<td>3.7 Exploring the Basic Properties of Rational Functions</td>
<td>42</td>
</tr>
<tr>
<td>3.8 Exploring the Basic Properties of Exponential Functions</td>
<td>47</td>
</tr>
</tbody>
</table>
3.9 Exploring the Basic Properties of Logarithms .......................... 51
3.10 Inverse Functions ................................................................. 56
3.11 Comparing the End Behavior of Power Functions and Exponential Functions ......................................................... 62
3.12 Comparing the End Behavior of Logarithms and Square Root . . 69
3.13 Global and Local Behavior of Power Functions ......................... 71
3.14 Left End Behavior of Power Functions ..................................... 75
3.15 End Behavior of Rational Functions ......................................... 78
3.16 Secant and Tangent Lines ......................................................... 82
3.17 The Derivative ............................................................... 86
3.18 The Derivative at Local Maxima and Minima .............................. 91
3.19 The Area Under the Curve ......................................................... 94
IV. CONCLUSION .......................................................... 99
BIBLIOGRAPHY .......................................................... 100
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Does this picture represent a function?</td>
<td>13</td>
</tr>
<tr>
<td>3.2</td>
<td>Does this picture represent a function?</td>
<td>14</td>
</tr>
<tr>
<td>3.3</td>
<td>Is this function One-to-One?</td>
<td>32</td>
</tr>
<tr>
<td>3.4</td>
<td>Is this function One-to-One?</td>
<td>33</td>
</tr>
<tr>
<td>3.5</td>
<td>Is this function Onto?</td>
<td>34</td>
</tr>
<tr>
<td>3.6</td>
<td>Is this function Onto?</td>
<td>35</td>
</tr>
<tr>
<td>3.7</td>
<td>Are these functions inverses of each other?</td>
<td>57</td>
</tr>
<tr>
<td>3.8</td>
<td>Are these functions inverses of each other?</td>
<td>58</td>
</tr>
<tr>
<td>3.9</td>
<td>Sum up the area of these rectangles.</td>
<td>95</td>
</tr>
<tr>
<td>3.10</td>
<td>Sum up the area of these rectangles.</td>
<td>96</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

There is currently a push in the state of Ohio to enhance the teaching of concepts foundational to Calculus in grades 8-12. In fact, the Ohio Department of Education sent out a call for grant proposals in the spring of 2005 concerning the development of programs that would focus on concepts foundational to Calculus. Ideas such as local behavior, global behavior, and rates of change which have traditionally been taught in Calculus to the upper level math students in their senior year of high school are now recommended to be introduced in late middle or early high school. In addition, technology makes possible the study of other concepts such as rates of growth, treating functions as families, transformations, and optimization in these courses. This thesis contains inquiry-based lessons that explore Calculus concepts that are suitable for a student taking a lower level course such as Algebra or Pre-Calculus. (Note that in mathematics, inquiry is often used as a replacement for the term problem solving.) It is important to note that a graphing calculator is essential when completing these lessons. Algebra students will not have many of the tools to do these lessons by hand, plus many of the lessons would be extremely computationally heavy without a calculator. Since the goals of these lessons are to explore and discover mathematical concepts, the calculator will also allow the students to use the time
that could have been spent on tedious computation to explore deeper concepts such as pattern recognition.

The need for inquiry-based lessons in schools was increased with the 2000 revision of *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (NCTM). In this publication, higher level thinking skills, such as problem solving, were stressed as essential ingredients for all mathematics courses [1]. Further, with the current advances in technology, calculators can now do much of the computational work for the students, freeing their resources for these higher level skills. The inquiry-based lessons created for this thesis aim to give students the opportunity to be involved in meaningful problem solving activities that will expose them to upper level mathematics, the use of technology, and working with others. There are many other objectives to completing these lessons. First, these lessons will aim to teach the students specific problem solving techniques along with the fact that they should attain deeper conceptual knowledge of the topics covered and mathematics in general. Also, the students will attain background knowledge and experience with performing their own experiments to test given conjectures as well as their own conjectures [2]. Further, the facilitators and students can use these lessons as exploratory experiences which can lead to students asking their own questions and seeking their own answers.

There is plenty of research supporting the idea of teaching Calculus concepts with the use of inquiry and technology. In the Gail Burrill et al. meta-analysis of calculator use, it was concluded that students who have access to technology are
more likely to be engaged in problem solving activities [3]. Thus, we feel that, if used properly, technology and inquiry can go hand-in-hand. In terms of the actual material the students will be seeing, Burrill et al. state that students who use calculators understand the concept of a function better than those who did not. Also, students with technology access have demonstrated stronger understanding of variables and improved ability to solve algebraic problems. They also state that the use of technology created an environment where students worked together more with each other and were active participants in their classes [3].

Chapter 2 of this thesis contains the literature review. Such topics as the effectiveness of inquiry-based lessons and calculator usage will be discussed. Chapter 3 consists of the lessons themselves, along with sample homework questions and a few assessment problems for each. Chapter 4 concludes this thesis with a brief summary of the work completed.
CHAPTER II
LITERATURE REVIEW

From my own past experience, it seems that both the integration of technology and teaching in an inquiry-based manner have been met with much resistance. According to Rebecca Reiff, the most likely reason behind this is that many of today’s teachers were neither taught with powerful calculators, nor in an inquiry-based manner. It should be no surprise that teachers tend to teach the way they were taught. And since science and mathematics have traditionally been lecture-based we have the current struggle between tradition and reform [4]. The remainder of this chapter will discuss both sides of the technology struggle and will dive deeper into the benefits of inquiry-based teaching. Research discussing the importance of such topics as functions, variables, and number sense (a few of the main topics studied in the lessons developed) will also be included.

2.1 Calculators

In the 2000 revision of *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) reiterated that technology is an essential tool for learning mathematics effectively. *The Standards* state that technology extends the math that can be taught and enhances student learning [1]. However,
as mentioned above, there has been, and still is, a heated debate among educators concerning the effectiveness of the use of calculators. This section will continue with potential benefits and downfalls of calculator usage.

There have been many studies that have supported the use of calculators in schools. Studies have supported the ideas that calculator usage can result in increased achievement and improved student attitudes [5]. Also, there have been claims that calculators can increase the role of inquiry and that students will become more comfortable with using new technologies in our technologically driven world [6]. A study conducted by Garfield Burke Jr. of Mississippi Valley State University concluded that, when used appropriately, the overall results of calculators as learning tools were positive and that powerful opportunities for many types of learning and instruction take place when they are used [7]. Another study conducted by Robin Pennington of Salem-Teikyo University showed a significant correlation between calculator use (including instruction on how to use it and allowing their use on the post-test) and increased post-test scores [8]. A study conducted on the results of the Third International Mathematics and Science Study (TIMSS) showed that calculators were beneficial when used for nonroutine purposes such as number concepts and during complex problem solving [9]. Pomerantz and Waits took more of a qualitative approach to the benefits of calculator usage. They state that calculators will reduce the time on computation which can lead to an increased amount of time on understanding concepts, discovery, and observing patterns. In fact, they claim that there is very little mathematical reasoning involved in performing computation in the first
place, therefore the students will not miss out on anything by having the calculator do the computations for them. They also state that students who would normally be bored with computation will no longer feel the need to give up when confronted with tedious computations. Pomerantz and Waits also claim that access to calculators allows the students to come up with multiple solution techniques, not just the techniques that can be done by hand [10]. Burrill et al. state that students who use graphing calculators in a curriculum that supports its use have better understanding of functions, variables, solving applied algebraic problems, and interpreting graphs than those students who did not use them. Also, they state that students with graphing calculators spend more time engaged in problem solving and are more flexible in their strategies than students who did not have them. They also state that when students use calculators frequently, they do in fact learn both the strengths and limitations of them [3].

There has also been a good deal of research citing potential downfalls to allowing students calculator access. First, it was found that there was a significant negative relationship between calculator use and scores on the TIMSS for Japanese students [9]. The same study found for American students that routine use of the calculator did not contribute to achievement [9]. While many proponents of calculator use state that computations are tedious work with little math involved that require rote memorization, Marshall states that well mastered routines are necessary to open the students’ minds towards problem solving and that the ability to store and recall information is an important skill for students to have [11]. Burke says that calculator
use can give students a false sense of confidence about their mathematics ability [7] and Linn feels that it is possible that the students will spend too much time trying to learn how to use the calculator instead of learning how to do the math [6]. Hunsaker states that calculator use will result in students having a lack of constructive methods and will prevent students from seeing mathematical structure. She also says that students tend to get the feeling that the calculator is always right and that the students, if they hit a wrong button and do not notice that they have done so, do not question the potentially very incorrect answer. Most importantly, she feels that mathematics is studied to teach discipline, thinking skills, and to expand the mind, all of which she feels calculator use prohibits [12].

It is important to note that it seems that most of the proponents of calculator use support it only in certain situations and many of the opponents have their feelings due to the possibility that the calculator will be improperly used. This issue was covered by Thompson when he states that when deciding on whether or not to allow calculators, the teacher needs to focus on the educational objectives for the activity [13]. Thompson gives a wonderful example of a lesson that should involve the calculator. He tells the students to write a five digit number using the digits 1, 2, 3, 4, and 8 each one time. He then tells the students to divide the number by 9 and to record the remainder. (Note that the remainder will always be zero.) He then tells the students to do this a few times using the same digits and to make a conjecture concerning any relationship. He then asks the students to explain this apparent phenomenon and to find another set of five digits that would result in the
same phenomenon [13]. This is a good calculator problem because the objective of the lesson is not long division practice, but pattern recognition. The calculator, in this case, will allow the students to experiment with more sets of 5 numbers and will give them more time to make and/or prove a conjecture. However, if the teacher felt that the students were having a difficult time understanding the concept of division and wanted them to practice some long division problems as well, then the calculator would be an inappropriate tool for this lesson. It should all come down to the learning objective for each lesson.

2.2 Inquiry

The literature in support of inquiry-based learning is large and getting larger. The Standards claim that, “Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking” [1]. Peressini states that since inquiry involves students making conjectures and sharing them with others, it follows that students learn academic courage and modesty [14]. Students also learn how to communicate their ideas to others, an essential tool for students [15]. Marrero likes inquiry because when it is used, questions are usually realistic and they become more sophisticated as more concepts are understood [16]. In reviewing data from the TIMSS, Linn states that one of the main reasons that the Japanese are ahead of the United States in mathematics is because their cur-
riculum offers more opportunities for open inquiry when compared to the American curriculum [6]. Jarrett states that there is strong evidence that inquiry-based learning enhances student performance and attitudes about mathematics, fosters literacy, and strengthens vocabulary and critical thinking skills [17]. She also feels that being immersed in a meaningful challenge can be one of the students’ most satisfying life experiences [18]. The Standards states that by learning problem solving in mathematics, students will acquire different ways of thinking, habits of persistence and curiosity, as well as confidence in new and unfamiliar situations that will serve them well inside and outside of the classroom. The Standards also states that engaging students in inquiry can give them a chance to solidify and extend their mathematical knowledge [1]. A study conducted by Hanna Salovaara gives evidence that students who are involved in inquiry-based learning have deeper level cognitive strategies such as self-monitoring, whereas students not involved in inquiry demonstrate low level strategies like memorization [19]. Chapko and Buchko claim there is a huge difference between traditional lecture-based and inquiry-based mathematics. They state that lecture-based math works well in the short term, but that if students do not understand why the computation works then they will eventually forget the material. However, they claim that inquiry-based mathematics focuses on understanding and conceptual development. Further, since students are often prompted to discuss their solutions to a small group or the entire class, errors in their solutions can be noticed by others, discussed, and eventually corrected [20]. Gearhart and Saxe state that teachers cannot effectively guide students toward deeper understanding of challeng-
In other words, inquiry helps the teachers help the students. Burrill et al. state that problem solving creates an environment where students work together more and are active participants in their classes [3]. A mathematics teacher named Steven Reinhart even goes as far to say that once he began to implement inquiry-based approaches in his classroom his students simply enjoyed his classes much more than they had previously [22].

2.3 Functions, Variables, and Number Sense

In these lessons, Calculus concepts such as the limit of a function and rates of growth will be studied. While the students are working with these new concepts, they will surely be developing their skills with variables, functions, and number sense. The great mathematician and educator Felix Klein felt that the concept of the function is the heart of mathematics because it is present wherever mathematical thought is used [23]. Grouws says that teaching math with a focus on number sense encourages students to become problem solvers and helps them realize that thinking is important in mathematics [5]. Edwards feels that the concepts of variables, functions, and properties of numbers are the three essential algebraic concepts that students should learn. He also states that, in his experience, students can have a very difficult time understanding basic algebraic notation [24]. Since each one of these lessons incorporates the ideas of variables and functions, students will get plenty of experience
working with the notation that is so often taken for granted by those who understand it.

It is important for the facilitator to realize that the students will gain much more than just the Calculus concepts while completing these lessons. Their basic algebra skills and knowledge of notation should greatly improve which will prepare them for more complicated topics such as Trigonometry or Calculus. Also, since these lessons are following an inquiry-based approach, the students will get experience working with others and explaining their thoughts. While it will be most convenient to complete the lessons in the order provided, one can pick and choose certain lessons as long as the students have the appropriate background knowledge.
CHAPTER III
LESSONS, HOMEWORKS, AND ASSESSMENTS

3.1 The Properties of Functions

3.1.1 Lesson

This lesson is designed to help the student understand the basic properties of a function. Such properties include the Domain, Range, and Codomain of a function.

**Definition** A function $f$ is a correspondence from a set $X$ to a set $Y$ where each element of $X$ has exactly one element from $Y$ associated with it. If $x_1$ is an element of $X$ that is associated with $y_1$ in $Y$, then we denote this as $f(x_1) = y_1$. In this case, we call $x_1$ an origin and $y_1$ its image under $f$. The set $X$ is called the Domain of $f$ and the set $Y$ is called the Codomain of $f$.

1. Observe Figure 3.1. Does this picture represent a function from $X$ to $Y$? Justify your answer by citing the definition of a function.

2. Now observe Figure 3.2. Does this picture represent a function from $X$ to $Y$? Justify your answer by citing the definition of a function.

3. Refer again to Figure 3.2. What are $f(A)$, $f(B)$, and $f(C)$? Explain.

4. It is often useful to put functions into algebraic expressions. Suppose that we know that $g$ is some function such that $g(1) = 2$, $g(2) = 3$, $g(3) = 4$, etc. Find
Figure 3.1: Does this picture represent a function?

an algebraic expression for \( g(x) \). In other words, if we plug any number \( x \) into the function \( g \), then what number will \( g(x) \) equal?

5. Let \( f(x) = 2x + 1 \). The table below can be very helpful when finding the value of a function at a certain \( x \) value. The first two are completed for you. Notice that we are just replacing the variable \( x \) with the constant values of the top row. Then compute on your own \( f(10) \) and \( f(15) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( f(1) = 2(1) + 1 = 3 )</td>
<td>( f(5) = 2(5) + 1 = 11 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Let’s try another one. Let \( g \) be a function such that \( g(x) = -x + 6 \). Complete the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>-5</th>
<th>2</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>( g(-1) = -(-1) + 6 = 7 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7. Let’s try one more example. Let $h$ be a function such that $h(x) = x^2 + 1$. Complete the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>3</th>
<th>5</th>
<th>$-7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>1</td>
<td>10</td>
<td>26</td>
<td>80</td>
</tr>
</tbody>
</table>

8. Using the your calculator’s Table tool, complete the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>13</td>
</tr>
</tbody>
</table>

9. Let $f(x) = x^2 + 1$. Change xmin to -5, xmax to 5, ymin to 0, and ymax to 20. Sketch the graph of the equation $y = x^2 + 1$. (The notation for this graphing window is $[-5, 5] \times [0, 20]$ and will be used frequently throughout these lessons.)

10. Notice that we stated that we graphed the equation $y = x^2 + 1$. What we did was let $y = f(x)$ for any given $x$. For the equation $y = x^2 + 1$ we are searching out
ordered pairs that make both sides of the equal sign equivalent expressions. Below, explain the distinction between the equation \( y = x^2 + 1 \) and the function \( f(x) = x^2 + 1 \).

11. Let \( g(x) = \frac{1}{x - 6} \). What is the Domain of \( g \)? Explain your answer algebraically and graphically.

12. Let \( f(x) = \sqrt{x} \). Plug in a few positive and a few negative numbers for \( x \) and compute \( f(x) \). From this, what can you conclude about the Domain of \( f \)? Explain your answer.

13. You also could have answered this question graphically. Graph the function \( f(x) = \sqrt{x} \) and explain how you can find the Domain of \( f \) from the graph.

14. The Range of a function is the set of all \( y \) values such that some \( x \) gets mapped to it. The Codomain of the function can then be considered to be any set that contains the Range. Refer back to Figure 3.2. What is the Range of the function represented by this picture? What is the Codomain of the function?

15. Sketch the graph of \( f(x) = \sin x \) below.

16. What is the Range of \( f \)? Explain why you think this.

17. From 16, name three different sets that could be the Codomain of \( f \). Explain why all three of these sets are legitimate Codomains.

18. Sketch the graph of \( g(x) = x^2 + 4 \) below. State the Range of \( g \) and then describe three sets that could be the Codomain of \( g \). Explain your answers.

19. List all of the definitions and properties that you learned in this lesson concerning functions.
3.1.2 Homework

Complete this homework after doing lesson 3.1.1.

1. Let \( f (x) = \sqrt{x + 3} \). Complete the table below using your calculator.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>5</th>
<th>13</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Sketch a graph of the function below. From this graph, find the Domain of \( f \). Explain your answer.

3. Support your answer in 2 by explaining algebraically why your Domain is correct. (Hint: What kind of number can one not take the square root of?)

4. From analyzing the graph in 2, find the Range of \( f \). Explain your answer.

5. Using your answer in 4, find three sets that could be the Codomain of \( f \). Explain why these sets are legitimate Codomains.

6. Let \( g(x) = \frac{2}{(x - 5)(x + 1)} \). Sketch the graph of \( g \) below.

7. From the graph in 6, find the Domain of \( g \). Explain your answer.

8. Support your answer in 7 by explaining algebraically why your Domain is correct.

9. By analyzing the graph in 6, find the Range of \( g \). Explain your answer.

10. From the Range of \( g \) found in 9, name two sets that could be the Codomain of \( g \). Explain why these sets are legitimate Codomains.

Extension. There is a special constant called \( e \) where \( e \approx 2.718 \). (The \( \approx \) symbol means approximately equal to.) \( e \) shows up often in mathematics. One time is in bank accounts. It turns out that if a bank gives a 5% rate of return compounded
continuously, then the amount of money you have in the account, \( A \), at any time, \( t \), in years, is

\[
A(t) = Pe^{0.05t},
\]

where \( P \) is the initial principal (the money you put in). Assume that you put $1000 into this account.

1. Sketch a graph of the function \( A \) below.
2. How much money will be in the account after 2 years? How about after 8 years? Explain.
3. Find the Domain of \( A \). Justify your answer.
4. Now find the Range of \( A \) from your answer in 3. Interpret your answer and explain why it is correct.

3.1.3 Assessment Questions

Sample assessment questions for lesson 3.1.1.

1. Let \( f(x) = \sqrt{x - 1} + 4 \). Complete the table below using your calculator.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>6</th>
<th>12</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Find the Domain of \( f \) algebraically and then confirm your answer graphically.
3. Find the Range of \( f \). Explain your answer.
4. Find three sets that could be the Codomain of \( f \). Explain why these are legitimate choices.
5. Let \( g(x) = \frac{2}{x - 8} \). Find the Domain of \( g \). Explain your answer.

6. Describe three real world situations where the concept of a function will be useful.

3.2 The Zeroes (Roots) of a Function

3.2.1 Lesson

This lesson is designed to help the student understand the concept of a zero of a function and explores different methods on how to find the zeroes of a given function. This lesson also explores the concept of when the graph of a function touches or crosses the \( x \)-axis at each zero.

**Definition** Let \( f \) be a function. If \( f(c) = 0 \), then \( c \) is called a zero of \( f \). Sometimes the word root is used instead of zero.

**Part 1: Different Methods for Finding Zeroes**

1. Consider the equation \( 3x - 6 = 0 \). State which \( x \) value satisfies the equation.

2. Considering your answer in 1, what is the zero of the function \( f(x) = 3x - 6 \)?

3. Consider the equation \( x^2 + x - 2 = 0 \). Factor the left side of the equation and state which \( x \) values satisfy the equation.

4. Now consider the function \( f(x) = x^2 + x - 2 \). Considering your answer to 3, what are the zeroes of \( f \)?
5. Describe how to find the zeroes of a function.

6. Now graph the function \( f(x) = x^2 + x - 2 \). Is there a way that you could have found the zeroes of \( f \) by doing this? Explain.

7. Graph the function \( g(x) = (x - 1)(x + 4)(x - 5) \). What are the zeroes of \( g \)? Justify your answer.

8. How could you have found the zeroes for the above function \( g \) simply by looking at the function?

9. Notice that it would be very difficult to factor the function \( g(x) = 2.5x^3 + 7x^2 - 4.2x - 12.9 \). Sketch the graph of \( g \) below.

10. From this graph, find the zeroes of \( g \). Explain your answers.

**Part 2: When does the Graph Cross or Touch the x-axis**

**Definition** Let \( f(x) \) be a function with a zero at \( x = c \). The *Multiplicity of the zero* \( c \) is how many times the term \((x - c)\) divides \( f(x)\).

11. Let \( f(x) = (x - 1)^2 (x + 4)^5 (x + 2) \). We know that 1, -4, and -2 are the zeroes of \( f \). Complete the table below by finding the Multiplicity of the zero -2.

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-4</td>
<td>5</td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>
12. As seen above, the multiplicity of a given zero $c$ is simply the exponent associated with the factor $(x - c)$. Let $g(x) = 7x^4 (x - 8)^5 (x + 2) (x - 6)^8$. Complete the following table.

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

13. Sketch the graphs of the functions $f(x) = x - 1$, $g(x) = (x - 1)^2$, $h(x) = (x - 1)^3$, and $i(x) = (x - 1)^4$. All of these functions have a zero at $x = 1$. Notice that two of the graphs cross the $x$-axis at $x = 1$ and two seem to touch and then bounce off the $x$-axis at $x = 1$. Which do which?

14. Now graph the function $g(x) = 7x^4 (x - 8)^5 (x + 2) (x - 6)^8$. Complete the table below.

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
<th>Cross or Touch</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. Now graph the function we looked at in 11, $f(x) = (x - 1)^2 (x + 4)^5 (x + 2)$. Then complete the table below.
16. From the tables in 14 and 15, can you make a guess as to which values of the multiplicity lead to a touch or a cross? Explain.

17. Let \( h(x) = (x + 1)^3 (x - 2)^5 (x + 3)^2 (x - 9)^4 (x - 1)^3 \). From your guess in 16, complete the following table without graphing the function. Explain why you chose touch or cross for each.

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
<th>Cross or Touch</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

18. State all new definitions and properties you learned in this lesson.

19. Explain all of the methods for finding zeroes of a function that you learned in this lesson.

Extension. Let \( f(x) = x^4 - 5x^3 + 3x^2 + 5x - 4 \). Sketch the graph of the function below. From that sketch, write \( f(x) \) in factored form. (For example, \( g(x) = \)
\((x - 1)^6 (x + 2)^3 (x + 4)^2\) is in factored form.) Explain your reasoning concerning where it touches and/or crosses.

### 3.2.2 Homework

Complete this homework after doing lesson 3.2.1.

1. Find the zeroes of \(f(x) = x^2 + 8x + 15\) by factoring. Confirm your answer graphically.

2. Find the zeroes of \(g(x) = x^2 + 4x - 12\) by factoring. Confirm your answer graphically.

3. Using any method you choose, find the zeroes of the function \(f(x) = x^3 - 6x^2 + 5x + 2\).

4. Using any method you choose, find the zeroes of the function \(g(x) = x^2 + 1\).

5. Let \(f(x) = (x + 1)^2 (x - 2)^3 (x - 6)^5 (x - 1)^8\). List the zeroes of \(f\) and state whether the graph touches or crosses the \(x\)-axis at each zero. Explain.

6. Find three different functions all having zeroes at -3, 0, 2, and 6.

**Extension.** Suppose the velocity of a moving particle follows the function \(V(t) = t^3 - 6t^2 + 9t\) where \(V\) is in meters per second and \(t\) is in seconds. At which time(s) is(are) the particle not moving? Explain your answer.

### 3.2.3 Assessment Questions

Sample assessment questions for lesson 3.2.1.

1. Find the zeroes of \(f(x) = x^2 + x - 6\) by factoring.

2. Using any method you choose, find the zeroes of \(g(x) = 2x^3 - 7x^2 + x + 4\).
3. Let \( h(x) = 2x^3(x - 2)^4(x - 7)^5(x + 1) \). Find all of the zeroes of \( h \) and state whether the graph touches or crosses the \( x \)-axis at each zero. Explain your answers.

4. Describe 3 real world situations where finding the zero of a function may be useful.

Extension. If a bowling ball falls from a 100 meter tall building, the distance that it travels downward in meters, \( s \), can be represented by the equation \( s = \frac{1}{2}(9.8)t^2 \), where \( t \) represents the time it has been falling in seconds. How long will it take for the ball to reach the ground assuming that it is dropped from the top of the building?

3.3 Solving Equations

3.3.1 Lesson

This lesson is designed to help the student understand how to solve equations algebraically, graphically, numerically, and by using the difference of two functions.

1. Solve the equation \( 2x - 5 = 3x - 8 \) algebraically.

2. Graph the functions \( f(x) = 2x - 5 \) and \( g(x) = 3x - 8 \) on the same window. Sketch the graph below.

3. At what point do \( f \) and \( g \) meet? Considering 1, does this surprise you? Explain.

4. Using a table, make tblStart = 2.8 and make \( \Delta \text{tbl} = 0.1 \). Complete the box below. Which \( x \) value makes \( y1 \) and \( y2 \) equal? Is this surprising? Explain.
<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = 2x - 5$</th>
<th>$y_2 = 3x - 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Set $h(x) = f(x) - g(x)$ and sketch the graph of $h$ below. What do you notice about $h$ at the $x$ value where $f$ and $g$ meet? Explain why this is happening.

6. Set $f(x) = x^2 - 3$ and set $g(x) = 12x + 10$. Sketch a graph of both functions on the window $[-10, 10] \times [-5, 100]$ and guess at which $x$ value(s) $f(x) = g(x)$.

7. Set $h(x) = f(x) - g(x)$ and graph $h$. After observing the behavior of $h$, does it appear that you may have missed an $x$ value where $f(x) = g(x)$? Explain.

8. Now choose a more appropriate window on which to graph $h$ and find all of the zeroes of $h$.

9. Above, you have seen 4 different ways to solve an equation. Namely, we have used algebra, graphs, tables, and the difference of two functions. Describe each method and discuss when you think each method might be best used.

10. Solve the equation $e^x = x^2$ using any combination of methods you choose. Be sure to show your work below.
11. Support why you think there are no more solutions to the equation in 10 than the one(s) you obtained.

12. Solve the equation \( x^3 - 2x + 1 = 2x^2 - 3 \) using any combination of methods that you choose. Be sure to show your work below.

13. Support why you think there are no more solutions to the equation in 12 than the one(s) you obtained.

14. Solve the equation \( x^3 + 2x - 10 = 2^x \) using any combination of methods that you choose. Be sure to show your work below. Be very careful!

15. Support why you think there are no more solutions to the equation in 14 than the one(s) you obtained.

16. List all of the definitions, properties, and methods for solving equations that you learned in this lesson.

Extension. Aaron and David each have a savings account. Aaron has $4,000 in his account and David has $3,000 in his. Aaron’s account is compounded quarterly at a rate of 2% per year. Thus, the amount of money Aaron has at any time \( t \) can be represented by \( A(t) = 4000 \left( 1 + \frac{0.02}{4} \right)^{4t} \), where \( t \) is in years. David’s account is getting 5% interest and is compounded continuously. Thus, the amount of money David has at any time \( t \) is \( D(t) = 3000e^{0.05t} \). If neither of them put any more money into their respective accounts, how long will it take for the amount of money in David’s account to surpass the amount in Aaron’s? Make sure you respond using the correct units! Which method(s) did you use?
3.3.2 Homework

Complete this homework after doing lesson 3.3.1.

1. Solve the equation $3x + 8 = x + 12$ algebraically.

2. Using 1, what point do you think the equations

\[
y = 3x + 8 \\
y = x + 12
\]

have in common? Explain your answer.

3. Solve the equation $5x = 3^x$. Use any method you choose and explain why there are no other solutions.

4. Solve the equation $4^x - 1 = x^2 + 3x$. Use any method you choose and explain why there are no other solutions.

5. Solve the equation $-x^2 + 1 = 2x + 2.001$. Use any method you choose and explain why there are no other solutions.

Extension. It has been found that houses in the Akron area are growing in value by about 4% per year. Thus, the value of a home bought today for $200,000 will be worth $V(t) = 200000(1.04)^t$, where $t$ is in years and $V$ is in dollars. How long will it take for the house to be worth $300,000 assuming that the value continues to increase 4% per year? Explain your answer.
3.3.3 Assessment Questions

Sample assessment questions for lesson 3.3.1.

1. Let \( f(x) = x^2 + 3x + 1 \) and let \( g(x) = 2^x - 1 \). Find all of the points that are on the graph of both \( f \) and \( g \). Explain why there are no other points in common.

2. Solve the equation \(-2x + 8 = 4x - 10\) algebraically.

3. Let \( f(x) = x^3 - 1 \) and let \( g(x) = -3x \). Find all of the zeroes of \( f(x) - g(x) \). Explain.

4. Describe 3 real world situations where solving equations would be useful.

Extension. I am thinking of a number. I can tell you that 20 less than the square of the number is the same value as 15 more than \(-2\) times the number. Given this information, is it possible to find the exact number that I am thinking of? Justify your response.

3.4 Solving Inequalities

3.4.1 Lesson

This lesson is designed to help the student understand how to solve an inequality graphically.

1. We will start by solving the inequality \( y \geq x^2 \). To start this problem, graph the equation \( y = x^2 \) and sketch the graph below.
2. Now choose 3 points on the “inside” of the parabola, 3 points on the “outside” of the parabola, and 3 points on the parabola. For example, (0, 1) would be a point inside, (3, 0) would be a point outside, and (4, 16) would be a point on the parabola.

Points Inside =

Points Outside =

Points On =

3. For all of the chosen points, how does the y value relate to the $x^2$ value? Explain this relationship.

4. From 3, how can the region where $y \geq x^2$ be characterized?

5. Shade in the region where $y \geq x^2$.

6. Now let’s try to solve the inequality $y \geq -x^2 - 1$. Start by sketching the graph of $y = -x^2 - 1$.

7. Now find 3 points inside the curve, 3 points outside of the curve, and 3 points on the curve.

Inside Points =

Outside Points =

Points On =
8. Follow the same procedure used above to find the region where \( y \geq -x^2 - 1 \). Shade in the region where the inequality holds and justify your response.

9. Explain in your own words how to solve an inequality. Discuss drawing a curve, picking points, and shading a region.

Extension 1. Solve the inequality \( y < x^3 \). (Note that \( y = x^3 \) will not satisfy \( y < x^3 \). We denote this by drawing the curve \( y < x^3 \) with a dotted line, not a solid line.)

Extension 2. Solve the inequality \( 2y + 10 > 2x^4 \).

Extension 3. Suppose that Sally starts up a lemonade stand. She knows that she makes a profit of $0.17 for each glass of lemonade she sells. If Sally wants to make a profit of at least $20.00, how many glasses of lemonade does she need to sell?

3.4.2 Homework

Complete this homework after doing lesson 3.4.1.

1. Find the region where \( y \geq 2x + 8 \).

2. Find the region where \( y < x^2 - 2x + 3 \).

3. Find the region where \( 2y + 6 > 3x + 10 \).

Extension. Find the region where \( y < 2x \). On a separate graph, find the region where \( y > x \). From these two regions, find the region where \( y < 2x \) AND \( y > x \). Explain why you chose the region that you did.
3.4.3 Assessment Questions

Sample assessment questions for lesson 3.4.1.

1. Find the region where $y > 3x + 1$.

2. Find the region where $2x + 4 < 2y$ AND $y < -2(x + 3)^2$.

3. Describe 3 real world situations where solving inequalities would be useful.

Extension. Suppose that you are the Chief Executive Officer (CEO) of Intel. Intel currently has $500 million in cash. However, you want to buy an up-and-coming chip company called Processors ‘R Us for $750 million. As the CEO of Intel, you know that your company makes a $95 profit on every processor it sells. How many processors will Intel need to sell to have enough cash to buy Processors ‘R Us?

3.5 One-to-One and Onto Functions

3.5.1 Lesson

This lesson is designed to help the student understand the concepts of One-to-One and Onto functions.

Recall the following definition:

**Definition** A function $f$ is a mapping from a set $X$ to a set $Y$ where each element of $X$ has exactly one element from $Y$ associated with it. If $x_1$ is an element of $X$ that is associated with $y_1$ in $Y$, then we denote this as $f(x_1) = y_1$. The set $X$ is called the **Domain of** $f$ and the set $Y$ is called the **Codomain of** $f$. 
This lesson discusses two types of special functions. The first that will be studied is a One-to-One function.

Note that if the Domain of a given function is not specified, then the Domain is assumed to be the largest set possible contained in the set of real numbers. The set of real numbers will oftentimes be denoted by the $\mathbb{R}$ symbol.

1. Let $f(x) = \sqrt{x}$, $g(x) = x + 1$, and $h(x) = -x^3$. Sketch a graph of each of the functions below. Also, on each graph draw a few horizontal lines that intersect that graph at least once.

2. Let $i(x) = x^4$, $j(x) = 3x^2$, and $k(x) = \sqrt{1-x^2}$. Sketch a graph of each of these functions below. Again, on each graph draw a few horizontal lines that intersect that graph at least once.

3. Do you notice anything interesting about the number of intersection points of each horizontal line with $f$, $g$, and $h$ that doesn’t always hold true for $i$, $j$, and $k$? Explain.

4. From your response in 3, is it possible to have two different $x$-values being mapped to the same $y$ value for $f$, $g$, and $h$? Relate your response to the number of intersection points with the horizontal lines.

Any function that has the property that any horizontal line will cross the graph at most once is called One-to-One. A more formal definition is given below.

**Definition** A function $f$ is called One-to-One if, for all $x_1$ and $x_2$ in the Domain of $f$ where $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. In other words, each $y$ value has at most one $x$ being mapped to it.
5. Is the function represented by the picture in Figure 3.3 a One-to-One function? Explain by citing the definition.

6. Is the function represented by the picture in Figure 3.4 a One-to-One function? Explain by citing the definition.

7. Let \( f(x) = x^2 \). Let the Domain of \( f \) be \( \mathbb{R} \) and the Codomain of \( f \) be \( \mathbb{R} \). Sketch a graph of \( f \) below.

7. Can you find \( x_1 \) and \( x_2 \) such that \( x_1 \neq x_2 \) but \( f(x_1) = f(x_2) \)? Explain.

8. Using the definition of a One-to-One function, is \( f \) One-to-One when the Domain is \( \mathbb{R} \)? Explain.

9. Now consider the function \( g(x) = x^2 \). However, let the Domain of \( g \) be all nonnegative real numbers, \( \{x|x \geq 0\} \), and let the Codomain of \( g \) be all nonnegative real numbers, \( \{y|y \geq 0\} \). Sketch the graph of the function below.
10. Can you find \(x_1\) and \(x_2\) such that \(x_1 \neq x_2\) but \(g(x_1) = g(x_2)\)? Explain.

11. Can you find any horizontal line that crosses the graph of the function more than once? Explain. (Note that this is commonly referred to as the Horizontal Line Test).

12. Is \(g\) a One-to-One function? Explain.

13. Notice that we have studied two functions, one of them being One-to-One and the other was not. However, both of the functions appeared to be the same since \(f(x) = x^2\) and \(g(x) = x^2\). Therefore, what must be specified when one is to determine if a function is One-to-One? Explain using the previous two examples.

14. Let \(h(x) = 2(x - 1)^4\). Let the Domain of \(h\) be \(\mathbb{R}\) and the Codomain of \(h\) be \(\mathbb{R}\). Sketch the graph of the function below.
15. Using either algebra or the Horizontal Line Test, explain why $h$ is not One-to-One.

16. Come up with a different domain that would make $h(x) = 2(x - 1)^4$ a One-to-One function. Explain why the function is now One-to-One.

We also want to study functions that are called *Onto*.

![Figure 3.5: Is this function Onto?](image)

17. Observe Figure 3.5 and Figure 3.6. Explain the relationship that exists between the Range and the Codomain of the function in Figure 3.6 that does not in Figure 3.5.

18. The function represented in Figure 3.6 is called *Onto*. In your own words, describe the property that must hold for a function to be Onto.

**Definition** A function $f$ is called *Onto* if the Codomain of $f$ is equal to the Range of $f$. 

34
19. Let \( f(x) = x^3 \) and let the Domain of \( f \) be \( \mathbb{R} \) and the Codomain be \( \mathbb{R} \). Sketch the graph of \( f \) below.

20. Does it appear that the Range of \( f \) is \( \mathbb{R} \)? In other words, are the Range and the Codomain the same for \( f \)? Explain.

21. Now let \( g(x) = x^3 \) and let the Domain of \( g \) be \( \{x | x \geq 0\} \) and the Codomain be \( \mathbb{R} \). Sketch the graph of \( g \) below.

22. Is \( g \) Onto? Cite the definition of Onto to justify your answer.

23. Notice that in the past two examples, one of the functions was Onto and one was not. This happened even though the functions appeared to be the same \( (f(x) = x^3 = g(x)) \). Therefore, what must be specified if one is to determine if a function is Onto? Explain using the previous two examples.
24. Let \( h(x) = x^2 \). Let the Domain of \( h \) be \( \mathbb{R} \) and let the Codomain of \( h \) be \( \mathbb{R} \). Sketch the graph of the function below.

25. Notice that \( h \) is not Onto. Explain why this is so.

26. Change either the Domain, the Codomain, or both to make the function \( h(x) = x^2 \) an Onto function. Explain why your answer works.

27. List all of the definitions and properties you learned in this lesson.

Extension 1. Let \( f(x) = (x - 4)^2 \). Come up with a Domain and a Codomain that would make \( f \) both One-to-One and Onto.

Extension 2. Let \( f(x) = 2x \). Explain why \( f \) is One-to-One no matter what Domain you choose. Generalize this concept to all functions where one of the following properties hold:

1) If \( x_1 < x_2 \), then \( f(x_1) < f(x_2) \)

2) If \( x_1 < x_2 \), then \( f(x_1) > f(x_2) \)

3.5.2 Homework

Complete this homework after doing lesson 3.5.1.

1. Let \( f(x) = 2x - 1 \). Let the Domain of \( f \) be \( \mathbb{R} \) and let the Codomain of \( f \) be \( \mathbb{R} \). Is \( f \) One-to-One? Explain.

2. Using \( f \) as in 1, is \( f \) Onto? Explain.

3. Give an example of a linear function that is not One-to-One. Explain why it is not One-to-One.
4. Let $g(x) = e^x$. Let the Domain of $g$ be $\mathbb{R}$ and let the Codomain be $\mathbb{R}$. Is $g$ One-to-One? If yes, explain why. If $g$ is not One-to-One, explain why and give a different Domain that would make $g$ One-to-One.

5. Using $g$ as in 4, is $g$ Onto? If $g$ is Onto, explain why. If $g$ is not Onto, explain why and give a different Codomain that would make $g$ be Onto.

6. Let $h(x) = 2x^2 + 3x - 5$. Find some Domain and Codomain that will make $h$ One-to-One and Onto. Explain your answer.

Extension. Come up with one example for each of the following:

- A function that is neither One-to-One nor Onto.
- A function that is One-to-One but not Onto.
- A function that is Onto but not One-to-One.
- A function that is both One-to-One and Onto.

Explain each of your responses along with the Domain and Codomain of each.

3.5.3 Assessment Questions

Sample assessment questions for lesson 3.5.1.

1. Let $f(x) = 4x$. Let the Domain of $f$ be $\{x \mid x \geq 2\}$. Find the Codomain that will make $f$ One-to-One and Onto. Explain your answer.

2. Let $g(x) = x^2 + 1$ with the Domain of $g$ being $\mathbb{R}$ and the Codomain of $g$ being $\mathbb{R}$. Is $g$ One-to-One, Onto, both, or neither. Explain.

Extension. Let $h$ be a quadratic function with the Domain of $h$ being $\mathbb{R}$. Will $h$ ever be One-to-One? Explain.
3.6 Exploring the Basic Properties of Polynomial Functions

3.6.1 Lesson

The objective of this lesson is to understand the basic properties of polynomial functions. Such properties include Domain, Range, end behavior, possible number of zeroes, and shapes.

Part 1: Domain

1. Consider \( f(x) = 3x^2 + 2x - 4 \) and \( g(x) = 5x^3 - 5x^2 + 21x - 4 \). Where are these functions defined? In other words, are there \( x \)-values for which either \( f \) or \( g \) is not defined?

2. What can you conclude about the Domain of any polynomial?

Part 2: Range and End Behavior

Definition The notation \( x \to \infty \) is spoken “As \( x \) approaches infinity” or “As \( x \) goes to infinity.” This refers to \( x \) getting larger and larger in an unbounded manner. \( x \to -\infty \) is spoken “As \( x \) approaches negative infinity” and refers to \( x \) going farther and farther to the left on the number line in an unbounded manner. The behavior of a function \( f \) as \( x \to \pm \infty \) is often referred to as the End Behavior of the function \( f \).

Definition The Degree of a polynomial is the value of the largest exponent. For example, \( f(x) = 3x^2 - 10 \) has degree of 2 and \( g(x) = 15x^3 - 12x + 20 \) has degree of 3.

3. Let \( f(x) = -x^3 \), \( g(x) = x^5 \), and \( h(x) = x^7 \). Describe the end behavior of all three as \( x \to \pm \infty \).
4. From the end behavior of \( f, g, \) and \( h \), find the Range of all three functions.

5. What do you notice about the degree of \( f, g, \) and \( h \)? How is this affecting the end behavior and Range of each? Explain by discussing what happens to the sign of any number that is raised to an odd power.

6. Let \( i(x) = x^2 \), \( j(x) = -x^4 \), and \( k(x) = -x^6 \). Describe the end behavior of all three as \( x \to \pm \infty \).

7. Find the Range of \( i, j, \) and \( k \).

8. Explain why the Ranges of \( i, j, \) and \( k \) were different than the Ranges of \( f, g, \) and \( h \). Discuss what happens to any number when it is raised to an even power.

9. First guess, and then check, the Range for each of the following functions: \( f(x) = x^3 + 2x - 1 \) and \( g(x) = -x^4 + 1 \). Did your answers agree with what you did above?

10. In your own words, describe the end behavior of a polynomial given the fact that it has even degree. Then describe the end behavior if the function has odd degree.

**Part 3: Possible Number of Zeroes**

11. Using any method you choose, find the number of different zeroes that each function below has. It does not matter *what* the zeroes are, we care about *how*
many different zeroes there are for each.

\[ f(x) = x^2 - 3 \]
\[ g(x) = x^2 + 2 \]
\[ h(x) = x^2 \]

12. Reflecting on your answer in 11, can you find a polynomial of degree two with more than two zeroes? Explain.

13. Using any method you choose, find the number of different zeroes that each function below has. It does not matter what the zeroes are, we care about how many different zeroes there are for each.

\[ f(x) = x^3 + 1 \]
\[ g(x) = x^3 - 3x^2 + 4 \]
\[ h(x) = x^3 + 2x^2 - x - 2 \]

14. Notice that all of the examples in 13 had at least one zero. Thinking about the end behavior of polynomials of odd degree, explain why all degree three polynomials will have at least one zero.

15. Can you find a polynomial of degree three with four or more zeroes? Explain your answer.

16. Consider a polynomial \( f \) of degree \( n \), where \( n \) is any integer. From what you have observed with polynomials of degree two and three, make a conjecture concerning the greatest number of zeroes that \( f \) can have.
Part 4: Shapes According to Degree

17. For polynomials of degree two, we have seen that there is either 0, 1, or 2 different zeroes. Draw a second degree polynomial, \( f \), with 1 zero and \( f(x) \to -\infty \) as \( x \to \pm \infty \).

18. Now draw a polynomial, \( f \), of degree two that has two different zeroes and \( f(x) \to \infty \) as \( x \to \pm \infty \).

19. We also saw that polynomials of degree three had either 1, 2, or 3 different zeroes. Draw a polynomial, \( f \), of degree three that has three different zeroes and \( f(x) \to -\infty \) as \( x \to \infty \). Be sure to give proper end behavior. Remember, this polynomial is of odd degree!

20. Draw a polynomial, \( f \), of degree three that has one zero and \( f(x) \to \infty \) as \( x \to \infty \). Be sure to give proper end behavior. Remember, this polynomial is of odd degree!

21. In your own words, describe the properties and definitions that you learned in this lesson about polynomials.

3.6.2 Homework

Complete this homework after doing lesson 3.6.1.

1. Let \( f(x) = 3x^2 + 2x - 1 \). Find the Domain and Range of \( f \). Explain your answers.

2. Let \( g(x) = 3x^3 - 7 \). Find the Domain and Range of \( g \). Explain your answers.
3. Describe the end behavior of \( h(x) = x^4 + 3x - 8 \) as \( x \to \pm \infty \).

4. Draw a polynomial of degree two with no zeroes that approaches \( \infty \) as \( x \to \pm \infty \).

Extension. Suppose that \( f \) is a polynomial of degree three. Suppose also that \( f \) has only two different zeroes. Describe the behavior of \( f \) as it approaches the two zeroes. In other words, at which zero will it touch and at which zero will it cross?

To help you, consider the two functions \( g(x) = 21x + 8x^2 + x^3 + 18 = (x + 2)(x + 3)^2 \)
and \( h(x) = -40x - 11x^2 - x^3 - 48 = -(x + 3)(x + 4)^2 \).

3.6.3 Assessment Questions

Sample assessment questions for lesson 3.6.1.

1. Find the Domain and Range of \( f(x) = 3x^3 - 2x + 5 \). Explain your answers.

2. Describe the end behavior of \( g(x) = 4x^6 + 5x - 7 \) as \( x \to \pm \infty \).

3. Describe the end behavior of \( h(x) = 4x^7 + 3x^6 - 100 \). Can you conclude anything about the number of zeroes of \( h \) simply by looking at the function without graphing it? Can you find the minimum number or maximum number of zeroes?

3.7 Exploring the Basic Properties of Rational Functions

3.7.1 Lesson

This lesson is designed to help the student understand the properties of rational functions. These properties include Domain, Range, zeroes, and asymptotes.
**Definition** A *Rational Function* is a function of the form \( f(x) = \frac{p(x)}{q(x)} \),
where \( p \) and \( q \) are polynomials.

**Part 1: Domain**

1. Let \( f(x) = \frac{(x - 1)(x + 3)}{(x - 2)(x - 8)} \). Are there any \( x \)-values for which \( f \) is not defined? (Hint: Remember we cannot divide by zero!) From this, find the Domain of \( f \).

2. Let \( g(x) = \frac{x}{(x + 1)^2(x - 5)} \). What is the Domain of \( g \)? Explain why the Domain of \( g \) is not all real numbers.

3. Let \( h(x) = \frac{x + 1}{x(x^2 + 4)} \). What is the Domain of \( h \)?

4. Explain why \( h \) in 3 has the same Domain as \( i(x) = \frac{x + 1}{x} \).

5. Let \( h(x) = \frac{p(x)}{q(x)} \), where \( p \) and \( q \) are both polynomials. What is the Domain of \( h \)? Explain by referring to the previous examples.

**Part 2: Zeroes**

6. Let \( f(x) = \frac{(x - 1)(x + 2)}{x^2} \). Find all of the zeroes of \( f \). Confirm your thoughts by sketching a graph of the function below.

7. Let \( g(x) = \frac{x(x - 2)^3(x + 4)^2}{x - 9} \). Find all of the zeroes of \( g \). Explain why \( g \) has these specific zeroes.

8. By now you may have a good idea of where a rational function has a zero. However, examine this next function. Let \( h(x) = \frac{(x + 1)(x - 3)}{(x + 1)} \). Explain why \( h \) does not have a zero at \( x = -1 \).
9. Let \( k(x) = \frac{p(x)}{q(x)} \) be a rational function. By examining the examples in 6, 7, and 8, describe the zeroes of \( k \). Is \( x_0 \) necessarily a zero of \( k \) simply because \( p(x_0) = 0 \)? Explain.

**Part 3: Vertical Asymptotes**

**Definition** The line \( x = c \) is a *Vertical Asymptote* of the function \( f \) if as \( x \) approaches some constant \( c \) from the left and the right, \( f(x) \) approaches either \( \infty \) or \( -\infty \).

10. Let \( f(x) = \frac{1}{x-2} \). Sketch a graph of \( f \) below.

11. Fill out the following tables:

<table>
<thead>
<tr>
<th></th>
<th>2.1</th>
<th>2.01</th>
<th>2.001</th>
<th>2.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>1.9999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

12. Describe the behavior of \( f(x) \) as \( x \) gets close to 2 from the right side of the number line.

13. Describe the behavior of \( f(x) \) as \( x \) gets close to 2 from the left side of the number line.

14. What can you conclude about the line \( x = 2 \) for the function \( f \)?

15. Let \( g(x) = \frac{x}{(x-1)^2} \). Sketch a graph of \( g \) below.
16. As was done in 11, complete two tables to support where \( g \) may have a vertical asymptote.

17. Find the vertical asymptote for \( g \).

18. What is the difference between \( f \) and \( g \) close to their respective vertical asymptotes? Explain why this difference is occurring.

19. Let’s solidify this idea a little more. Graph the following functions on their own set of axes: \( f_1(x) = \frac{1}{x} \), \( f_2(x) = \frac{1}{x^2} \), \( f_3(x) = \frac{1}{x^3} \), \( f_4(x) = \frac{1}{x^4} \), and \( f_5(x) = \frac{1}{x^5} \).

20. Which of the above functions approached the same infinity as \( x \to 0 \)? Which ones approached different infinities as \( x \to 0 \)?

21. Make a conclusion concerning why a function will approach the same infinity from both sides. Do likewise for approaching different infinities from both sides.

22. Without graphing the function, find the vertical asymptotes of \( f(x) = \frac{10}{x^6(x-1)^3(x+5)^2(x-2)} \). For each vertical asymptote \( x = c \), tell whether \( f(x) \) will approach the same or different infinities as \( x \to c \).

23. State all definitions and properties that you learned in this lesson.

3.7.2 Homework

Complete this homework after doing lesson 3.7.1.

1. Let \( f(x) = \frac{2}{(x+1)(x+4)^2} \). Find the Domain of \( f \). Explain your answer.
2. Let \( g(x) = \frac{2x}{x(x-1)^3} \). Find the Domain of \( g \). Explain your answer. Be careful!

3. Let \( f(x) = \frac{(x-1)(x-2)}{2(x+4)(x+6)^2} \). Find all of the zeroes of \( f \). Explain your answer.

4. Let \( g(x) = \frac{(x-1)(x+5)}{x(x-1)^3} \). Find all of the zeroes of \( g \). Explain your answer. Be careful!

5. Let \( f(x) = \frac{1}{(x-5)(x-4)^2} \). Find all of the vertical asymptotes of \( f \). State whether \( f(x) \) approaches the same infinity or different infinities as \( x \) approaches each asymptote.

Extension. Sketch a graph of the function \( g(x) = \frac{x(x+1)}{x(x-5)} \). Explain why \( g \) has a vertical asymptote at the line \( x = 5 \), but not one at the line \( x = 0 \).

3.7.3 Assessment Questions

Sample assessment questions for lesson 3.7.1.

1. Let \( f(x) = \frac{(x+4)(x-5)^2}{x(x-3)(x-5)^2} \). Find the Domain of \( f \).

2. Find all of the zeroes of \( f \).

3. Let \( g(x) = \frac{x(2x-3)}{(x-6)^3(x-2)^5(x+1)^2} \). Find all of the vertical asymptotes of \( g \) and tell whether \( g(x) \) approaches the same infinity or different infinities as \( x \) approaches each asymptote. Justify each response.

Extension. Create a rational function with zeroes at \( x = 5 \) and \( x = 6 \) and vertical asymptotes at \( x = 0 \) and \( x = 8 \).
3.8 Exploring the Basic Properties of Exponential Functions

3.8.1 Lesson

This lesson is designed to help the student understand the properties of exponential functions. These properties include shapes of the functions, Domain, Range, zeroes, and end behavior. The concepts of One-to-One and Onto will also be studied.

**Definition** An *Exponential Function* is a function of the form \( f(x) = a^x \), where \( a \) is a real number and \( a > 0 \) and \( a \neq 1 \).

**Part 1: Exploring Different Shapes**

1. We will begin by looking at the graphs of 4 different exponential functions. Set \( f(x) = 1.5^x \), \( g(x) = 2^x \), \( h(x) = \left(\frac{1}{2}\right)^x \), and \( j(x) = \left(\frac{2}{3}\right)^x \). Sketch the graph of the functions on the window \([-5,5] \times [0,32]\). Label each of the functions along the way. Do all of the functions have something in common? Why is this so?

2. Notice that two of the graphs you see get bigger as \( x \to \infty \) and two of them get smaller as \( x \to \infty \). Which ones do which?

3. Do you notice anything about the \( a \) values of the functions that get bigger as \( x \to \infty \) compared to the \( a \) values of the functions that get smaller as \( x \to \infty \)? Why is this happening?

4. Make a conjecture about which \( a \) values will make the function \( f(x) = a^x \) get bigger as \( x \to \infty \). Likewise, make a conjecture about which \( a \) values will make the function \( f(x) = a^x \) get smaller as \( x \to \infty \). Feel free to graph other functions to convince yourself.
Part 2: Domain of Exponential Functions and One-to-One and Onto

5. Go back to the graph that you saw in 1. Does it seem like each exponential function is defined at each $x$ value?

6. Can you think of any reason why any of these functions would not be defined at a given $x$ value outside of the Domain $[-5, 5]$? Explain.

7. Make a conjecture about the Domain of any exponential function.

8. From your observations, are exponential functions One-to-One? Justify by citing the definition of a One-to-One function.

9. Assuming that the Domain of any exponential function $f$ is $\mathbb{R}$, give the corresponding Codomain which would make $f$ Onto. Also, give an example of a Codomain which would make $f$ not Onto. Explain both of your responses.

Part 3: Range, Zeroes, Horizontal Asymptotes, and End Behavior of Exponential Functions

10. Let $f(x) = 2^x$. Record the following values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-10000$</th>
<th>$-100$</th>
<th>$100$</th>
<th>$1000$</th>
<th>$10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. As $x \to \infty$, what is $f(x)$ approaching? Explain.

12. As $x \to -\infty$, what is $f(x)$ approaching? Explain.

13. Now graph $f$ on the window $[-10, 10] \times [-1200, 1200]$. Notice that you can only see a little piece of the graph. What happened to the left side of the graph?
graph?  In other words, when $x < 0$, what numerical value does $f(x)$ seem to be approaching?

14. From your observations in 10, 12, and 13, will $f(x)$ ever reach the functional value of zero? Explain.

15. From your observations in 11, 12, and 14, make a conjecture about the Range of $f(x) = 2^x$.

16. From 15, will the function $f(x) = 2^x$ have any zeroes? Explain.

17. Do you think the Range of $g(x) = 3^x$ will be the same as the Range of $f(x) = 2^x$? How about $h(x) = \left(\frac{1}{2}\right)^x$? Will either of these function have a zero? Justify your answer.

18. State what you discovered in 12 concerning the function $f(x) = 2^x$.

Definition  When $f(x)$ gets closer and closer to the horizontal line $y = c$ as $x \to \pm\infty$, we call the line $y = c$ a Horizontal Asymptote of the function $f$.

19. Does the function $f(x) = 2^x$ have a horizontal asymptote as $x \to -\infty$? Explain.

20. The line $y = 0$ is a horizontal asymptote for the function $h(x) = \left(\frac{1}{2}\right)^x$. Is this due to the way the function behaves as $x \to \infty$ or as $x \to -\infty$? Explain.


22. State in your own words all of the new terminology and properties that you learned in this lesson.
3.8.2 Homework

Complete this homework after doing lesson 3.8.1.

1. Let \( f(x) = 4^x \) and let \( g(x) = \left(\frac{1}{3}\right)^x \). Describe the similarities/differences of the end behavior of the two functions. Mention the behavior as \( x \to \infty \) and as \( x \to -\infty \).

2. Find all horizontal asymptotes of \( f \) and \( g \). Explain your answers using words or a picture.

3. Find the Domain and Range of \( f \) and \( g \). Explain your answers.

4. Does either \( f \) or \( g \) have a zero? Explain.

5. Let \( h(x) = 4^x - 1 \). Notice that this is \( f(x) - 1 \) from the problems 1 - 4. Describe the end behavior of \( h \). Mention the behavior as \( x \to \infty \) and as \( x \to -\infty \).

6. Does \( h \) have a horizontal asymptote? Explain.

7. Notice that the horizontal asymptote of \( h \) was different than the asymptote for \( f \) and \( g \). Explain why this is so.

8. Find the Domain and Range of \( h \). Explain why the Range of \( h \) is different than that of \( f \) and \( g \).

9. Does \( h \) have a zero? If so, find it. If not, explain why.

Extension. A colony of 10 bacteria grows following the function \( N(t) = 10e^{0.1t} \), where \( t \) is in days and \( N \) represents the number of bacteria at any time.

1. As \( t \to \infty \), how does \( N \) behave? Explain.

2. What is the Domain of the function \( N \)? Explain.
3. What is the Range of the function \( N \)? Interpret this Range in terms of time \( t \).

4. How many days will it take for the colony to have 25 bacteria? Explain.

3.8.3 Assessment Questions

Sample assessment questions for lesson 3.8.1.

1. Let \( f(x) = 5^x \) and let \( g(x) = \left( \frac{1}{5} \right)^x \). Describe the Domain, Range, shape, zeroes, and end behavior of each function as \( x \to -\infty \) and as \( x \to \infty \).

2. Let \( h(x) = e^x + 5 \). Find the Range of \( h \). Explain your answer by referring to the end behavior of \( h \).

3. Find the horizontal asymptote of the function \( f(x) = 2^x - 8 \). Explain your answer.

Extension. Let \( f(x) = 3^{x-2} \). Describe the Domain, Range, zeroes, and end behavior of \( f \). (Hint: Use properties of exponents to simplify \( f \) so that \( f(x) = c(3^x) \) for some constant \( c \).)

3.9 Exploring the Basic Properties of Logarithms

3.9.1 Lesson

This lesson is designed to help the student understand the properties of logarithmic functions. These properties include shapes of the functions, Domain, Range, zeroes, asymptotes, and end behavior.
Definition We say that \( y = \log_a x \) is a logarithm if and only if \( a^y = x \) and \( a > 1 \). Likewise, you can write a logarithm as a function \( f(x) = \log_a x \) if and only if \( a^{f(x)} = x \).

We start this lesson setting the base \( a \) equal to the number \( e \approx 2.718 \). When this is the case, we say that \( \log_e x = \ln x \) (read “Natural Log of \( x \”\).

Part 1: Studying the Natural Logarithm

1. Graph the function \( f(x) = \ln x \) on the window \([-5, 5] \times [-5, 5]\) and sketch the graph below.

2. Is \( f \) defined at \( x = 0 \)? If so, what is \( f(0) \)? If not, explain why \( f \) is not defined at this point using the definition.

3. From 1 and 2, what do you think the Domain of \( f \) is. Explain.

4. Does \( f \) have a zero? At which \( x \) value? Justify your answer.

5. Let’s now look at the end behavior of \( f \). Record the following values:

<table>
<thead>
<tr>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00001</td>
</tr>
<tr>
<td>0.00001</td>
</tr>
<tr>
<td>0.000001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00001</td>
</tr>
<tr>
<td>0.00001</td>
</tr>
<tr>
<td>0.000001</td>
</tr>
</tbody>
</table>

6. As \( x \to 0 \), what is \( f(x) \) approaching? As \( x \to \infty \) what is \( f(x) \) approaching? Explain.

7. From 6 and the graph of \( f(x) = \ln x \), what is the Range of \( f \)?
Notice that you discovered in 6 that as \( x \to 0 \), \( f(x) \to -\infty \). In other words, as you get closer and closer to the line \( x = 0 \), \( f(x) \) becomes unbounded in the negative \( y \) direction. Thus, the line \( x = 0 \) is a Vertical Asymptote of the function \( f \).

Recall the definition:

**Definition** The line \( x = c \) is a Vertical Asymptote of a function \( f \) if, as \( x \) gets close to \( c \), \( f(x) \to \pm \infty \).

**Part 2: Studying other Logarithmic Functions**

8. Now let’s look at a few other logarithms. Graph the functions \( g(x) = \log_2 x \) and \( h(x) = \log_5 x \) along with \( f(x) = \ln x \) on the window \([-5, 5] \times [-5, 5] \) and sketch the graphs below. Note that you will not have a \( \log_2 \) or \( \log_5 \) button. However, we can use a theorem that tells us that \( \log_b a = \frac{\ln a}{\ln b} \). Knowing this, simplify \( \log_2 x \) and \( \log_5 x \) and then graph.

9. Do the three graphs have the same kind of shape? Explain.

10. Based on the graph and what you learned about \( f(x) = \ln x \), what do you think the Domains of \( g \) and \( h \) are? Explain.

11. Record the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.00001</th>
<th>0.000001</th>
<th>0.0000001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g(x) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h(x) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
12. What do you think $g$ and $h$ are approaching as $x \to 0$? How about when $x \to \infty$? Explain.

13. Based on 11 and 12, what do you think the Range of $g$ is? How about the Range of $h$? Explain.

14. Do $g$ and $h$ have a vertical asymptote? If so, what is the equation of the asymptote?

15. Do $g$ and $h$ have the same zero that $f(x) = \ln x$ had? How do you know?

Part 3: Conclusions about Logarithms

16. From what you have seen in this lesson, make a conclusion concerning the Domain of any logarithmic function.

17. From what you have seen in this lesson, make a conclusion concerning the Range of any logarithmic function.

18. From what you have seen in this lesson, make a conclusion concerning the zeroes of any logarithmic function.

19. From what you have seen in this lesson, make a conclusion concerning the vertical asymptotes of a logarithmic function.
20. From what you have seen in this lesson, make a conclusion concerning what happens to a logarithmic function as $x \to \infty$.

21. State all new properties and definitions that you learned in this lesson.

3.9.2 Homework

Complete this homework after doing lesson 3.9.1.

1. Let $f(x) = \ln(x - 1)$. Find the Domain and Range of $f$.

2. Does $f$ have a zero? If yes, find it and explain why it is a zero. If not, explain why the function does not have a zero.

3. As $x \to \infty$, what is $f(x)$ approaching? Explain.

4. Does $f$ have a vertical asymptote? Is so, where? Explain.

5. Let $g(x) = 3 + \log_{5}(x + 2)$. Find the Domain and Range of $g$.

6. Does $g$ have a zero? If yes, find it and explain why it is a zero. If not, explain why the function does not have a zero.

7. As $x \to \infty$, what is $g(x)$ approaching? Explain.

8. Does $g$ have a vertical asymptote? If so, where? Explain.

Extension. Let $h(x) = \log_{a}k(x)$, where $a$ is some constant and $k$ is some function. Make a conjecture concerning where the vertical asymptote(s) of $h$ will be. Explain using examples that you have seen thus far.

3.9.3 Assessment Questions

Sample assessment questions for lesson 3.9.1.

1. Find the Domain and Range of the function $f(x) = 2 + \log_{3}(x + 4)$. 
2. Does the function \( f(x) = 2 + \log_3(x + 4) \) have a zero? If yes, find it. If not, explain why \( f \) does not have a zero.

3. Describe the end behavior of \( f \) as \( x \to \infty \) and as \( x \to -4 \).

4. Find the vertical asymptote of the function \( f(x) = 2 + \log_3(x + 4) \).

Explain your answer.

3.10 Inverse Functions

3.10.1 Lesson

This lesson is designed to help the student understand the definition of an inverse function. This lesson will also give the student methods to determine if two functions are inverse functions of each other.

1. Let \( f(x) = 2x + 4 \) and let \( g(x) = \frac{x}{2} - 2 \). Complete the tables below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -8 )</th>
<th>( -2 )</th>
<th>( 4 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -12 )</th>
<th>( 0 )</th>
<th>( 12 )</th>
<th>( 24 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Examine the tables in 1. Describe the relationship between the functions \( f \) and \( g \)?

3. Suppose \( f(a) = b \). Compute \( g(b) \). Suppose \( g(c) = d \). Compute \( f(d) \).

Explain your answers in terms of the relationship found in 2.

Due to this relationship, \( f \) and \( g \) are called \textit{Inverse Functions}.
Definition Let $f$ be a function with Domain $X$ and Codomain $Y$ where $f(x_0) = y_0$. If $g$ is a function with Domain $Y$ and Codomain $X$ where $g(y_0) = x_0$, then $g$ is called the Inverse Function of $f$.

4. Look at Figure 3.7. Is the function represented by Figure 3.7B the inverse function of the function represented in Figure 3.7B? Justify your answer by citing the definition.

5. Now look at Figure 3.8. Are the functions represented in 3.8A and 3.8B inverses of each other? Explain your answer by citing the definition.

6. Consider a function $f$ that has the points $(0,1), (2,6), (3,5),$ and $(10,-2)$ on its graph. Find 4 points on the graph of $f^{-1}$.

7. Let’s look at another pair of functions that are inverses. Let $f(x) = \frac{x + 4}{2x - 6}$ and let $f^{-1}(x) = \frac{6x + 4}{2x - 1}$. Below, choose any four $x$ values (labeled $a, b, c,$
and \(d\) below) you would like and evaluate \(f(x)\) for each of them. Then, evaluate \(f^{-1}(f(a))\), \(f^{-1}(f(b))\), etc.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(a =)</th>
<th>(b =)</th>
<th>(c =)</th>
<th>(d =)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(f^{-1}(x))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(f^{-1}(f(x)))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(f(f^{-1}(x)))</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Describe what happens in the table in 7. Use this to explain in your own words what inverse functions do when composed with each other.

9. Now sketch a graph of \(f^{-1}(f(x))\) below. Give another function that has the same graph. Explain.

10. Let’s look at another example. Let \(f(x) = e^x\) and let \(g(x) = \ln x\). Again, choose any four \(x\) values and complete the table below.
11. From this table, what can you conclude about \( f(x) = e^x \) and \( g(x) = \ln x \). Explain.

12. Sketch a graph of \( f \) and \( g \) on the same set of axes below. About which line are the two functions symmetric?

13. Let’s look back at our first pair of inverse functions. That is, let \( f(x) = 2x + 4 \) and let \( g(x) = \frac{x}{2} - 2 \). Graph these two functions on the same set of axes. Are these functions symmetric about the same line? Explain.

14. List two different methods you can use to determine if two functions are inverses of each other.

15. List all definitions and properties that you learned in this lesson.

Extension 1. Let \( f(x) = x^2 \) where the Domain of \( f \) is \( \mathbb{R} \) and the Codomain of \( f \) is \( \mathbb{R} \). Let \( g(x) = \sqrt{x} \). Complete the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -5 )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g(f(x)) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Are the values in rows 1 and 3 of the table in Extension 1 always equal? For which values are they always equal? (Hint: Look at the sign of each \( x \) value.) Explain what is happening.
3. Now consider \( f(x) = x^2 \). However, let the Domain of \( f \) be \( \{x| x \geq 0\} \) and let the Codomain of \( f \) be \( \{y| y \geq 0\} \). Let \( g(x) = \sqrt{x} \) where the Domain of \( g \) is \( \{x| x \geq 0\} \) and the Codomain of \( g \) is \( \{y| y \geq 0\} \). Are these functions inverses? Explain by using a table.

| \( x \) | \( a = \) | \( b = \) | \( c = \) | \( d = \) |
|-----|-----|-----|-----|
| \( f(x) \) | \( \) | \( \) | \( \) | \( \) |
| \( g(f(x)) \) | \( \) | \( \) | \( \) | \( \) |

Recall the following definitions.

**Definition** A function \( f \) is called *One-to-One* if, for all \( x_1 \) and \( x_2 \) in the Domain of \( f \) where \( x_1 \neq x_2 \), \( f(x_1) \neq f(x_2) \). In other words, each \( y \) value has only one \( x \) being mapped to it. We can also say a function is One-to-One if it passes the *Horizontal Line Test*.

**Definition** A function \( f \) is called *Onto* is the Codomain of \( f \) is equal to the Range of \( f \).

4. Look again to Extension 3. When we restricted the Domain and Codomain of \( f \), we were able to find an inverse function. Explain why \( f \) must be One-to-One to have an inverse function. What goes wrong if \( f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \)? Look carefully at the definitions of a Function, Inverse Functions, and One-to-One for assistance.
5. Explain why \( f \) must be Onto to have an inverse function. What goes wrong if \( f \) is not Onto? Look carefully at the definitions of a Function, Inverse Functions, and Onto for assistance.

6. Refer back to a pair of inverse functions that we studied, \( f(x) = 2x + 4 \) and \( g(x) = \frac{x}{2} - 2 \). For \( f \), set \( f(x) = y \) and solve for \( x \) in terms of \( y \). What do you notice about this new equation?

7. Considering Extension 6, explain how to find the inverse function of any linear function.

3.10.2 Homework

Complete this homework after doing lesson 3.10.1.

1. Let \( f \) and \( g \) be inverse functions. If \( f(a) = b \), then compute \( g(b) \). Explain your answer.

2. Let \( f(x) = \frac{x + 2}{2x - 5} \). The inverse function of \( f \) is one of the following functions:

\[
g(x) = \frac{6x - 8}{2x - 3}
\]

\[
h(x) = \frac{5x + 2}{2x - 1}
\]

\[
k(x) = \frac{2x - 5}{4x + 6}
\]

Using any method you like, determine which function is the inverse function of \( f \). Support your answer.
3. In the space below, sketch a graph of the function \( f(x) = 2x - 4 \). After doing this, sketch a graph of \( f^{-1} \). Recall that there is a special symmetry between \( f \) and \( f^{-1} \).

Extension. Let \( f(x) = (x - 2)^2 \). Let \( g(x) = \sqrt{x} + 2 \). Find the Domain and Codomain of \( f \) so that \( g \) is the inverse function of \( f \). Explain your answers.

3.10.3 Assessment Questions

Sample assessment questions for lesson 3.10.1.

1. Let \( f \) and \( g \) be inverse functions. Suppose all of the following are true:

\[
\begin{align*}
  f(2) &= 6 \\
  f(4) &= 5 \\
  f^{-1}(1) &= 0 \\
  f^{-1}(3) &= 10
\end{align*}
\]

Compute \( f^{-1}(6) \), \( f^{-1}(5) \), \( f(0) \), and \( f(10) \). Explain your answers.

2. Sketch the graph of the functions \( f(x) = 3x + 2 \) and \( g(x) = x - 1 \). Are these functions inverses of each other? Explain by referring to your graph.

3.11 Comparing the End Behavior of Power Functions and Exponential Functions

3.11.1 Lesson

This lesson is designed to help the student compare the end behavior of two different types of functions, Power and Exponential. The definition of each type of function
Definition A Power Function is a function of the form \( f(x) = x^a \), where \( a \) is a constant integer greater than or equal to 1.

Definition An Exponential Function is a function of the form \( f(x) = a^x \), where \( a \) is a constant greater than 1. (Note that most books also allow \( 0 < a < 1 \). However, these exponential functions behave very differently from when \( a > 1 \), so we will not consider them in this lesson.)

Since each of these functions requires the use of some constant \( a \), let’s fix some \( a > 1 \) to get an idea of how these functions behave as \( x \to \infty \). We arbitrarily choose \( a = 2 \) for the next two examples.

1. Let \( f(x) = x^2 \), a power function with \( a = 2 \). Using your calculator, record the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. As \( x \to \infty \), what is happening to \( f(x) \)? Do you think that all power functions will work this way? Explain.

3. Now let \( g(x) = 2^x \), an exponential function with \( a = 2 \). Using your calculator, record the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. As \( x \to \infty \), what is happening to \( g(x) \)? Do you think that all exponential functions will work this way? Explain.
5. From your conjectures in 2 and 4, what is similar between the end behavior of power and exponential functions?

Just because all of the above functions go to infinity as \( x \to \infty \) does not mean that they grow in the same manner. One of these may still grow faster than the other one. Let’s try to figure out if there is a difference between the two rates of growth.

6. Consider the functions \( f_1(x) = x^2 \), \( f_2(x) = x^5 \), \( f_3(x) = x^{10} \), and \( g(x) = 2^x \). Record the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^{10} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2^x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. As \( x \to \infty \), which function grows the fastest?

8. Do you think that a power function with a larger exponent than 10 may grow faster than \( 2^x \)? Try it and record your results. (Note that you should not try anything bigger than \( x^{100} \) for this example. Remember that a calculator cannot represent numbers larger than a certain value!)

9. Make a conjecture concerning the rate of growth of an exponential function compared to a power function.
10. Let’s now consider the graphs of these functions. Graph the functions $f_1(x) = x^2$ and $g(x) = 2^x$ on the window $[0, 10] \times [0, 1200]$. Sketch the graphs below.

11. When $x$ gets large, what is the relationship between the two graphs? Which of the functions seems to be growing faster? What aspect(s) of the graphs are you looking at to determine this?

Instead of looking at large tables of recorded values (like in 6) to determine which functions grow faster than others, we can look at values of the quotient of two functions to determine which of them grows faster.

12. Let $f(x) = \frac{2^x}{x^2}$. Note that this is a power function divided by an exponential function. Record the following values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13. As $x \to \infty$, what appears to be happening to $f(x)$?

14. Using 12 and 13, which do you think grows faster as $x \to \infty$, $x^2$ or $2^x$? Justify your response.

15. Was your answer in 14 surprising, or were you expecting the results you obtained? Explain.

16. Consider the function $g(x) = \frac{x^2}{2^x}$. (Note that this is the reciprocal of $f(x) = \frac{2^x}{x^2}$, the function with which you just worked.) Record the following values:
17. As \( x \to \infty \), what appears to be happening to \( g(x) \)?

18. Using 16 and 17, which do you think grows faster as \( x \to \infty \), \( x^2 \) or \( 2^x \)?

Why?

19. Did you expect this result? Explain.

20. Let’s look at another example of this. Let \( f(x) = \frac{3^x}{x^{10}} \) and \( g(x) = \frac{x^{10}}{3^x} \).

Record the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

21. As \( x \to \infty \), what appears to be happening to \( f(x) \)? How about \( g(x) \)?

22. Using 20 and 21, which do you think grows faster as \( x \to \infty \), \( x^{10} \) or \( 3^x \)?

Why?

23. Was this expected? Explain.

24. State all definitions and properties you learned in this lesson.

Extension 1. Thinking about what you have just learned, consider the functions \( g(x) = \frac{2^x}{x^{1000000}} \) and \( h(x) = \frac{x^{7000000000000000}}{5^x} \). As \( x \to \infty \), what do you think \( g(x) \) will do? How about \( h(x) \)? Explain your reasoning. (Hint: Don’t even bother plugging values into your calculator!).
Extension 2. If \( f(x) \) grows faster than \( g(x) \) as \( x \to \infty \), then what does \( \frac{f(x)}{g(x)} \) approach as \( x \to \infty \)? How about \( \frac{g(x)}{f(x)} \)? Explain.

Extension 3. If \( \frac{t(x)}{h(x)} \to \infty \) as \( x \to \infty \), then which of the functions grows faster as \( x \to \infty \)? What about if \( \frac{t(x)}{h(x)} \to 0 \) as \( x \to \infty \)? Explain.

3.11.2 Homework

Complete this homework after doing lesson 3.11.1.

1. Let \( f(x) = e^x \) and let \( g(x) = x^{10} \). Let \( h(x) = \frac{g(x)}{f(x)} \). As \( x \to \infty \), describe the behavior of \( h(x) \).

2. From your answer in 1, what do you know about the growth of \( f(x) \) compared with the growth of \( g(x) \) as \( x \to \infty \).

3. Let \( k(x) = \frac{f(x)}{g(x)} = \frac{e^x}{x^{10}} \). As \( x \to \infty \), describe the behavior of \( k(x) \).

4. Judging by your answers in 2 and 3, what seems to be true about the growth of exponential and power functions?

5. Now let \( m(x) = e^x \) and let \( n(x) = x^{10} + 1000000 \). Describe the end behavior of \( p(x) = \frac{n(x)}{m(x)} \).

6. Describe the similarities and differences of the end behavior of \( h(x) = \frac{x^{10}}{e^x} \) and \( p(x) = \frac{x^{10} + 1000000}{e^x} \) as \( x \to \infty \).

7. Explain why adding 1000000 to the numerator of \( h(x) \) did not change the end behavior of \( p(x) \) as \( x \to \infty \).

Extension. Suppose that you have the choice of two bank accounts. If we
put in $100, Bank Account 1 grows at a rate following the function

\[ B_1(t) = 100e^{0.08t}, \]

where \( t \) is in years and \( B_1(t) \) represents the amount of money in the account at time \( t \). However, Bank Account 2 grows at a rate following the function

\[ B_2(t) = 100\left(1 + \frac{0.08}{2}\right)^{2t}, \]

where \( t \) is in years and \( B_2(t) \) represents the amount of money in the account at time \( t \).

Which account would you choose to put your money in? Justify your response by discussing the end behavior of \( B_1 \) and \( B_2 \).

3.11.3 Assessment Questions

Sample assessment questions for lesson 3.11.1.

1. Let \( f(x) = 2^x \) and \( g(x) = x^6 \). Describe the end behavior of \( h(x) = \frac{f(x)}{g(x)} \) as \( x \to \infty \).

2. Describe the end behavior of \( k(x) = \frac{g(x)}{f(x)} \) as \( x \to \infty \).

3. From 1 and 2, which function would you say “grows the fastest” between \( f \) and \( g \). Explain your choice.

4. Describe 3 real world situations where knowing “what grows faster” may be useful.
3.12 Comparing the End Behavior of Logarithms and Square Root

3.12.1 Lesson

This lesson is designed to help the student compare the end behavior of logarithmic functions and the square root function.

**Part 1: Comparing the Natural Log and the Square Root**

1. Sketch the graphs of \( f(x) = \ln x \) and \( g(x) = \sqrt{x} \) on the window \([-2, 2] \times [-2, 2] \).

2. What similarities do you notice about the functions on this window? What differences do you see?

3. As \( x \to 1 \), what is \( f(x) \) approaching? How about \( g(x) \)? How did you come up with your answer?

4. Consider the function \( h(x) = \frac{\ln x}{\sqrt{x}} \). As \( x \to \infty \), what is \( h(x) \) approaching? How did you come up with your answer?

5. From your response in 4, which function grows faster as \( x \to \infty \), \( \ln x \) or \( \sqrt{x} \)? Why do you think this?

**Part 2: Comparing other Logarithms to the Square Root**

6. Sketch the graphs of \( g(x) = \sqrt{x} \) and \( h(x) = \log_{1.5} x \) on the window \([-10, 10] \times [-10, 10] \).

7. From looking at the graph, which function do you think will grow faster as \( x \to \infty \), \( g(x) = \sqrt{x} \) or \( h(x) = \log_{1.5} x \)? Explain.
8. Now test your conjecture in 7. Using any method you would like, determine which function grows faster as $x \to \infty$? Explain your process.


10. Consider the function $r(x) = \frac{\sqrt{x}}{\log_{1.5} x}$. As $x \to \infty$, what will $r(x)$ approach? (Hint: Use 8 to help with this question.)

11. Let $s(x) = \log_a x$ ($a > 1$) and let $g(x) = \sqrt{x}$. From what you have seen in this lesson, which function do you think will grow faster as $x \to \infty$, $\log_a x$ or $\sqrt{x}$? Explain.

12. Using 11, what will $t(x) = \frac{\log_a x}{\sqrt{x}}$ approach as $x \to \infty$? How about $u(x) = \frac{\sqrt{x}}{\log_a x}$? Explain.

13. State all of the definitions and properties that you learned in this lesson.

3.12.2 Homework

Complete this homework after doing lesson 3.12.1.

1. Let $f(x) = \frac{\sqrt{x}}{\ln x + 100}$. Describe the end behavior of $f$ as $x \to \infty$.

2. Let $g(x) = \frac{\ln x + 100}{\sqrt{x}}$. Describe the end behavior of $g$ as $x \to \infty$.

3. From your answers in 1 and 2, which grows faster as $x \to \infty$, $\ln x + 100$ or $\sqrt{x}$? Explain.

4. Describe the end behavior of $h(x) = 500g(x) = \frac{500 (\ln x + 100)}{\sqrt{x}}$ as $x \to \infty$. Describe in terms of the end behavior of $g(x)$.
Extension. Let \( f(x) = a \ln x + b \), where \( a, b \geq 1 \). Let \( g(x) = \sqrt{x} \). Do you have enough information about both functions to know which grows faster as \( x \to \infty \)? Explain your answer by giving at least 4 examples.

3.12.3 Assessment Questions

Sample assessment questions for lesson 3.12.1.

1. Let \( f(x) = 2 \ln x \) and let \( g(x) = \sqrt{x} \). Which function grows faster as \( x \to \infty \)? Justify your response.

   Extension 1. Does the function \( f(x) = \frac{\ln x}{\sqrt{x}} + 1 \) have a horizontal asymptote as \( x \to \infty \)? If yes, find it and explain how you know. If no, explain why.

   Extension 2. Describe the end behavior of \( g(x) = \ln x - \sqrt{x} \) as \( x \to \infty \).

   Explain your answer in terms of the end behavior of \( \ln x \) and \( \sqrt{x} \).

3.13 Global and Local Behavior of Power Functions

3.13.1 Lesson

This lesson is designed to help the student compare the global and local behavior of different power functions.

Definition A Power Function is a function of the form \( f(x) = x^a \), where \( a \) is a constant integer greater than or equal to 1.

Part 1: Global Behavior

1. We will begin by comparing the functions \( x, x^2, x^3, \) and \( x^4 \). Record the values below:
2. List the functions in order from fastest growth to slowest growth. Do you notice any relationship between the ordering and the exponent in each function?

3. Now that we have looked at this concept numerically, let’s look at this idea graphically. We want to graph these four functions from \([0, 10] \times [0, 500]\). Sketch what you see below, labeling each function along the way.

4. As \(x \rightarrow \infty\), list the functions from fastest growth to slowest growth as seen in the graph. Do you notice any relationship to this order and the exponent in the function?

5. Is the order for the graphical method similar to the order found when looking at the functions numerically?

6. As \(x \rightarrow \infty\), make a conjecture concerning the rates of growth of power functions related to the exponent.

7. Without doing any graphing or numerical work, list \(x^{1000}\), \(x^{500}\), and \(x^{2000}\) from fastest growth to slowest growth as \(x \rightarrow \infty\).
Part 2: Local Behavior

The second part of this lesson will explore the behavior of different power functions on the Domain \([0, 1]\).

8. We will start by analyzing the same functions as in Part 1. Record the following values:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1/3</th>
<th>1/2</th>
<th>2/3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x^2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x^3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x^4)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. What do you notice about all of the functional values at \(x = 0\)? How about at \(x = 1\)?

10. Compare the behavior of all of the functions at the \(x\) values 1/3, 1/2, and 2/3.

11. Now graph the four functions on the window \([0, 1] \times [0, 1]\) and sketch it below, labeling each function along the way.

12. Does the graph confirm the thoughts you had concerning the behavior of the functions on the Domain \([0, 1]\)? Explain.

13. Make a conjecture concerning the behavior of power functions on the Domain \([0, 1]\).

14. Compare and contrast the local behavior of power functions to the global behavior of power functions.
Extension. Suppose you wanted to go sled riding and the hill follows the behavior of a power function (the top is (1, 1) and the bottom is (0, 0)) Would you prefer to go down the hill of \( f(x) = x \), \( g(x) = x^5 \), or \( h(x) = x^{100} \)? Explain. Note that there is no wrong answer! You get to choose!

3.13.2 Homework

Complete this homework after doing lesson 3.13.1.

1. Let \( f(x) = x^3 \) and let \( g(x) = x^5 \). As \( x \to \infty \), which function grows the fastest? Justify your answer.

2. Now let \( f(x) = 100x^3 \) and let \( g(x) = x^5 \). As \( x \to \infty \), which function grows the fastest? Justify your answer.

3. Now let \( f(x) = 500x^3 \) and let \( g(x) = x^5 \). As \( x \to \infty \), which function grows the fastest? Justify your answer.

4. From your answers in 2 and 3, make a conjecture concerning the rates of growth of \( f(x) = Ax^3 \) and \( g(x) = x^5 \) where \( A \) is any constant.

5. On the interval \((0, 1)\), sketch a graph of the functions \( f(x) = x^2 \), \( g(x) = x^5 \), and \( h(x) = x^{10} \).

6. From the graph in 5, describe which function is on top, which is in the middle, and which is on the bottom. Explain this order.

7. Compare and contrast the Global Behavior (as \( x \to \infty \)) and the Local Behavior (when \( x \) is in \((0, 1)\)) of power functions. Refer to the size of the exponent in your explanation.
Extension. Let \( f(x) = \frac{2x^4}{6x^8} \) and let \( g(x) = \frac{3x^7}{12x} \). Describe the end behavior of each function as \( x \to \infty \). Explain why each function has its particular end behavior.

3.13.3 Assessment Questions

Sample assessment questions for lesson 3.13.1.

1. Compare and contrast the end behavior of \( x^2 \), \( x^4 \), and \( x^9 \) as \( x \to \infty \).

2. Let \( f(x) = \frac{5x^2}{2x^6} \). Describe the end behavior of \( f \) as \( x \to \infty \) in terms of the rates of growth of \( 5x^2 \) and \( 2x^6 \).

3. Without doing any calculations, which value is the greatest of the following? Explain your answer.

\[
\left( \frac{1}{2} \right)^3 \\
\left( \frac{1}{2} \right)^8 \\
\left( \frac{1}{2} \right)^{12}
\]

3.14 Left End Behavior of Power Functions

3.14.1 Lesson

This lesson is designed to help the student understand the behavior of a power function as \( x \to -\infty \).

1. Using your calculator, record the following values:
2. How do you describe the behavior of these functions as $x \to -\infty$?

3. Do you notice a relationship between the value of the exponent and its behavior as $x \to -\infty$. What is this relationship?

4. Explain why the relationship in 3 exists. In other words, explain why some functions approach $+\infty$ and why some functions approach $-\infty$.

5. As $x \to -\infty$, which of the functions discussed approaches $-\infty$ the fastest? Explain.

6. As $x \to -\infty$, which of the functions discussed approaches $+\infty$ the fastest. Explain.

7. Compare and contrast the end behavior of power functions as $x \to -\infty$ to when $x \to \infty$ (You analyzed power function behavior as $x \to \infty$ in Global and Local Behavior of Power Functions).
3.14.2 Homework

Complete this homework after doing lesson 3.14.1.

1. Let \( f(x) = 2x^5 \) and let \( g(x) = -4x^6 \). Describe the end behavior of each as \( x \to -\infty \).

2. Why is \( g(x) = -4x^6 \) approaching \(-\infty\) as \( x \to -\infty \) even though the exponent is even? Explain.

3. Which function, \( f \) or \( g \), is going towards \(-\infty\) faster as \( x \to -\infty \). Justify your response.

4. Let \( f(x) = 3x^3 \) and let \( g(x) = 2x^4 \). Describe the end behavior of each as \( x \to \pm\infty \).

5. Let \( h(x) = \frac{f(x)}{g(x)} = \frac{3x^3}{2x^4} \). Describe the end behavior of \( h(x) \) as \( x \to \infty \) and as \( x \to -\infty \).

6. Let \( k(x) = \frac{g(x)}{f(x)} = \frac{2x^4}{3x^3} \). Describe the end behavior of \( k(x) \) as \( x \to \infty \) and as \( x \to -\infty \).

3.14.3 Assessment Questions


1. Describe the end behavior of \( f(x) = \frac{2x^5}{x^3} \) as \( x \to -\infty \).

2. Using your response in 1, describe the end behavior of \( -f(x) = -\frac{2x^5}{x^4} \).

3. Describe the end behavior of \( h(x) = \frac{6x^5}{3x^8} \) as \( x \to -\infty \).

4. Using your response in 3, describe the end behavior of \( -h(x) = -\frac{6x^5}{3x^8} \).
3.15 End Behavior of Rational Functions

3.15.1 Lesson

This lesson is designed to help the student understand the end behavior of a rational function.

**Definition** A function $r$ is called a *Rational Function* if $r(x) = \frac{p(x)}{q(x)}$, where $p$ and $q$ are both polynomials.

**Definition** The *Degree* of a polynomial is the value of the largest exponent. For example, $f(x) = 3x^2 - 10$ has degree equal to 2 and $g(x) = 15x^3 - 12x + 20$ has degree equal to 3.

**Part 1: Degree of Numerator equal to Degree of Denominator**

1. Let $f(x) = \frac{5x^2 - 10x + 6}{2x^2 + 15x + 3}$. Graph $f$ on the window $[-10, 10] \times [-10, 10]$ and sketch the graph below. Does this graph give you a good idea of the global behavior of $f$? Explain.

2. Now graph $f$ on the window $[-20, 20] \times [-10, 10]$ and sketch the graph below. Does this graph give you a better idea of the global behavior of $f$ than the graph in 1? Make a conjecture about the end behavior of $f$ by citing your sketch.

3. Graph $f$ on the window $[-500, 500] \times [-10, 10]$ and sketch the graph below.

4. What appears to be happening to the graph as $x \to \infty$? How about as $x \to -\infty$? Explain.
5. Recall that a *horizontal asymptote* is a horizontal line that a function gets closer and closer to as $x \to \infty$ or $-\infty$. Does $f$ seem to have a horizontal asymptote for when $x \to \infty$? How about when $x \to -\infty$? Explain.

6. Justify your thoughts from 5 by completing the tables below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1000000$</th>
<th>$-100000$</th>
<th>$-10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>10000</th>
<th>100000</th>
<th>1000000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Do the tables from 6 support your thoughts on the end behavior of $f$? Explain.

8. Recall we let $f(x) = \frac{5x^2 - 10x + 6}{2x^2 + 15x + 3}$. You have also described the end behavior of $f$ and found the horizontal asymptote. Does the value of the horizontal asymptote have a relationship with the coefficients of any of the terms of $f$? (Hint: Look closely at the coefficients of the $x^2$ terms.)

9. Let’s look at another example to see if this relationship described in 8 will occur again. Let $g(x) = \frac{5x^2 + 100x - 200}{2x^2 - 137x - 400}$. Graph $g$ on the window $[-500, 500] \times [-10, 10]$ and sketch the graph below. In your sketch, include the graph of $f$.

10. Does it appear that $g$ has the same horizontal asymptote as $f$? Explain.

11. Confirm your thoughts from 10 by completing the following tables.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1000000$</th>
<th>$-100000$</th>
<th>$-10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
12. Do the tables in 11 support your thoughts on the end behavior of $g$? Explain.

13. When comparing the functions $f$ and $g$, did the coefficients of the $x^1$ term or the constant term change the end behavior of the function? Explain.

14. For constants $A, B, C, D, E$, and $F$, $(A \neq 0$ and $D \neq 0)$ make a conjecture about the end behavior of $h(x) = \frac{Ax^2 + Bx + C}{Dx^2 + Ex + F}$.

15. Using the same methods you used above (graphically and by using a table of values), find the end behavior (as $x \to \infty$ and $x \to -\infty$) of the function

$$f(x) = \frac{4x^3 + 6x^2 - 9}{3x^3 + 20x + 34}.$$ 

16. Did you expect this end behavior? Explain.

17. If $h(x) = \frac{Ax^n + Bx^{n-1} + \cdots + Cx + D}{Ex^n + Fx^{n-1} + \cdots + Gx + H}$, what will $h(x)$ approach as $x \to \infty$? Explain.

**Part 2: Degree of Numerator less than Degree of Denominator**

18. Now graph the function $g(x) = \frac{7x^4 + 15x^3 + 2x}{2x^5 - 17x^3 - 14}$ and notice that as $x \to \pm\infty$, $g \to \frac{7}{2}$. Why did this occur? Does this go against any relationship you noticed about end behavior of rational functions in Part 1? Explain.

19. Graph the function $h(x) = \frac{2x^2 + 5}{6x^3 + 3x^2 - 8}$. As $x \to \pm\infty$, what does $h(x)$ seem to be approaching? Explain.
20. Let \( r(x) = \frac{p(x)}{q(x)} \) and let the degree of \( p \) be less than the degree of \( q \). As \( x \to \pm\infty \), what will \( r(x) \) approach? Explain.

**Part 3: Degree of Numerator greater than Degree of Denominator**

21. Graph the function \( f(x) = \frac{2x^2 - 1}{15x + 29} \). Describe the end behavior of \( f \).

22. Graph the function \( g(x) = \frac{3x^5 + 7x - 12}{23x^2} \). Describe the end behavior of \( g \).

23. Let \( r(x) = \frac{p(x)}{q(x)} \) and let the degree of \( p \) be greater than the degree of \( q \). As \( x \to \pm\infty \), what will \( r(x) \) approach? Explain.

24. Being as complete as possible, make a conjecture concerning the end behavior of a rational function in terms of the degree of the numerator and denominator.

### 3.15.2 Homework

Complete this homework after doing lesson 3.15.1.

1. Describe the end behavior of \( f(x) = \frac{2x^2 + 3x}{5x^2 + 100} \) as \( x \to \pm\infty \). How do you know this?

2. Does \( f \) from 1 have a horizontal asymptote? Explain.

3. Let \( g(x) = \frac{2x^2 + 3x}{5x^2 + 100} + 5 = f(x) + 5 \). Find the horizontal asymptote of \( g \) and explain your reasoning.

4. Let \( h(x) = \frac{3x^5 - 12}{4x^3 - 12x^2 + x} \). Describe the end behavior of \( h(x) \) as \( x \to \pm\infty \).

5. Does \( h \) have a horizontal asymptote? Explain how you know.
6. Let \( f(x) = \frac{5x^3 - 1}{7x^9 + 6x + 1} \). Describe the end behavior of \( f(x) \) as \( x \to \pm \infty \).

7. Does \( f \) from 6 have a horizontal asymptote? Explain how you know.

8. Let \( g(x) = \frac{5x^3 - 1}{7x^9 + 6x + 1} - 20 = f(x) - 20 \). Does \( g \) have a horizontal asymptote? Explain your reasoning.

3.15.3 Assessment Questions

Sample assessment questions for lesson 3.15.1.

1. Let \( f(x) = \frac{4x^3 + 2x^2}{100x} \). Describe the end behavior of \( f(x) \) as \( x \to \pm \infty \).

2. Let \( g(x) = \frac{4x^5 - 2x}{17x^7} \). Describe the end behavior of \( g(x) \) as \( x \to \pm \infty \).

Does \( g \) have a horizontal asymptote? Explain why or why not.

3. Let \( h(x) = \frac{3x^2 - 9}{5x^2 - 10x} \). Describe the end behavior of \( h(x) \) as \( x \to \pm \infty \).

Does \( h \) have a horizontal asymptote? Explain why or why not.

3.16 Secant and Tangent Lines

3.16.1 Lesson

This lesson is designed to help the student understand how one finds the tangent line to the graph of a function at a point by finding secant lines.

1. In the space below, sketch the graph the function \( f(x) = x^3 - 2x - 3 \) on the window \([-3, 3] \times [-10, 10]\).

2. For the function \( f \), compute \( f(-2) \) and \( f(2) \).

3. Now plot the two points on the graph that you drew in 1 and draw a line segment that has each point as an end point.
4. The segment that you just drew is called a Secant Line of the function $f$. A Secant Line is simply the segment connecting any two different points on the graph of $f$. Compute the slope of the secant line that you just created. Recall that slope is “Rise over Run” or $\frac{y_1 - y_2}{x_1 - x_2}$.

5. Now compute $f(-1)$ and $f(1)$. Plot these two points on your graph and draw the corresponding secant line.

6. Now compute the slope of this new secant line.

7. Compute $f\left(\frac{-1}{2}\right)$ and $f\left(\frac{1}{2}\right)$. Find the slope of the corresponding secant line.

8. Now compute $f\left(\frac{-1}{5}\right)$ and $f\left(\frac{1}{5}\right)$, $f\left(\frac{-1}{10}\right)$ and $f\left(\frac{1}{10}\right)$, and $f\left(-\frac{1}{20}\right)$ and $f\left(\frac{1}{20}\right)$. For each pair, find the slope of each of the corresponding secant lines and complete the table below.

<table>
<thead>
<tr>
<th>Leftmost $x$</th>
<th>Rightmost $x$</th>
<th>Slope of the Secant Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$2$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\frac{-5}{1}$</td>
<td>$\frac{5}{1}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{-1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{20}$</td>
<td>$\frac{1}{20}$</td>
<td></td>
</tr>
</tbody>
</table>

9. If we continued this process, what would our $x$ values be approaching? Explain.
10. As our $x$ values approach this value, do the different slopes of the secant lines seem to be approaching any particular value? Explain by examining the table in 8.

11. Draw a line with slope $-2$ through the point $(0, -3)$ on your graph.

The line that you just drew is called the Tangent Line of the function $f$ at the point $(0, -3)$. The Tangent Line is a very important concept in Calculus and other upper level mathematics. Let’s go about finding another tangent line to a different function at some point.

12. Let $g(x) = \frac{x^4}{2} - 6x + 10$. Graph $g$ on the window $[-3, 3] \times [-20, 20]$ and sketch the graph below. We will now find the tangent line through the point $(1, 4.5)$.

13. Following the method used above, complete the table below using the function $g$.

<table>
<thead>
<tr>
<th>Leftmost $x$</th>
<th>Rightmost $x$</th>
<th>Slope of the Secant Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$3$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$2$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$3$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>$\frac{5}{4}$</td>
<td></td>
</tr>
<tr>
<td>$9$</td>
<td>$11$</td>
<td></td>
</tr>
<tr>
<td>$10$</td>
<td>$10$</td>
<td></td>
</tr>
<tr>
<td>$99$</td>
<td>$101$</td>
<td></td>
</tr>
<tr>
<td>$100$</td>
<td>$100$</td>
<td></td>
</tr>
</tbody>
</table>
14. If we continued this process, what value would our $x$ values be approaching?

15. As our $x$ values approach this value, do the slopes of the secant lines seem to be approaching any particular value? Explain by citing the table in 13.

16. Thus, what seems to be the slope of the tangent line through the point $(1, 4.5)$ of the function $g$?

17. Let $h(x) = \ln(x^2)$. Using the method seen above, create a list of points approaching $x = 2$, find the slopes of the secant lines, and make a conjecture about the slope of the tangent line at $(2, h(2))$.

18. If given a function $f$, describe in your own words how to find the slope of the tangent line through a given point of the graph of $f$.

3.16.2 Homework

Complete this homework after doing lesson 3.16.1.

1. Let $f(x) = x^3 - 2x + 1$. Sketch a graph of $f$ below.

2. Draw the secant lines from $(1, 0)$ to $(2, 5)$ and from $(-1, 2)$ to $(0, 1)$ on the graph in 1. Compute the slope of each secant line.

3. Let $g(x) = 2x^2 - 1$. Determine the slope of the tangent line of $f$ at the point $(0, -1)$ by using the method learned in this lesson.

4. Let $h$ be a function. If the slope of the secant line from $(x_0, h(x_0))$ to $(x_1, h(x_1))$ is negative, what does that describe about the behavior of the function from $x = x_0$ to $x = x_1$. Explain with words and a picture.
3.16.3 Assessment Questions

Sample assessment questions for lesson 3.16.1.

1. Let \( f(x) = x^3 - 2 \). Find the slope of the secant line from \((0, -2)\) to \((3, 25)\).

2. Let \( g(x) = 2x^2 - 2x + 5 \). Find the slope of the tangent line of \( g \) at the point \((-1, 9)\).

3. Let \( h(x) = x^2 \). Find the slope of the tangent line of \( f \) at the point \((2, 4)\).

3.17 The Derivative

3.17.1 Lesson

This lesson is designed to help the student understand the concept of a derivative of a function at a certain point.

1. Suppose that Brett is running a 100 meter dash. It turns out that the total distance he has run follows the function \( f(t) = 5[(t + 1) \ln(t + 1) - t] \) where \( t \) represents the number of seconds. For example, after 1 second, Brett has traveled \( f(1) = 5[(2) \ln 2 - 1] = 1.93 \) meters and after 3 seconds he has run \( f(3) = 5[(4) \ln 4 - 3] = 12.72 \) meters. Sketch the graph of this function below on the window \([0, 13] \times [0, 100]\).

2. Draw the secant line containing the points \((2, f(2))\) and \((8, f(8))\). Draw this secant line on the graph above and find the slope of this secant line. We call
the slope of that secant line “The average rate of change of \( f \) from 2 to 8.” Explain why we would describe the slope of the secant line this way. Hint: If Brett runs \( f(8) - f(2) \) meters in \( 8 - 2 \) seconds then, on average, how far does he run for every second that elapses?

3. Now draw the secant line containing the points \((4, f(4))\) and \((6, f(6))\). Draw this secant line on the graph above and find the slope of this secant line.

4. As you may have guessed, we call the slope of the secant line found in 3 “The average rate of change of \( f \) from 4 to 6.” Explain why we would describe the slope of this secant line this way.

5. Since Brett runs the \( f(6) - f(4) \) meters in \( 6 - 4 = 2 \) seconds then, on average, how far does he run for every second that elapses during that time interval?

6. Now draw the secant line containing the points \((4.5, f(4.5))\) and \((5.5, f(5.5))\). Draw this secant line on the graph and find the slope of this secant line.

7. Since Brett runs the \( f(5.5) - f(4.5) \) meters in \( 5.5 - 4.5 = 1 \) second then, on average, how far does he run for every second that elapses during that time interval?

8. Now draw the secant line containing the points \((4.75, f(4.75))\) and \((5.25, f(5.25))\). Draw this secant line on the graph and find the slope of this secant line.

9. Record all of the slopes of the secant lines below. Notice that as we approach \( t = 5 \), the slopes of the secant lines are approaching some value. Thus, what do you think the slope of the tangent line at \( t = 5 \) will be? Explain why you
think this with respect to the secant line values.

<table>
<thead>
<tr>
<th>Leftmost $x$</th>
<th>Rightmost $x$</th>
<th>Slope of the Secant Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>5.5</td>
<td></td>
</tr>
<tr>
<td>4.75</td>
<td>5.25</td>
<td></td>
</tr>
</tbody>
</table>

10. Explain why the slope of the tangent line at $t = 5$ would be called “The instantaneous rate of change of $f$ at $t = 5.”

11. As you noticed, the average rate of change described the average rate of speed during a given time interval. Knowing this, explain why the slope of the tangent line would represent the speed of Brett at $t = 5$.

**Definition** For a given function $f$ and point $(c, f(c))$, the Derivative of $f$ at $x = c$ is the slope of the tangent line through the point $(c, f(c))$.

Let’s find the derivative at a different point of a different function. Let $g(x) = x^3 + 1$. Let’s find the derivative of $g$ at $x = 1$. (Note that the notation for this is $g'(1).$)

12. Complete the table below. As $x \to 1$, what do the average rates of change seem to be approaching? From this, evaluate $g'(1)$. Explain your reasoning.
13. On your own, evaluate $g'(2)$ and $g'(3)$ by using the same method we used above.

14. In your own words, list all properties and definitions that you learned in this lesson.

3.17.2 Homework

Complete this homework after doing lesson 3.17.1.

1. Let $f(x) = 3x - 2$. Compute $f'(1)$ by filling out the table below

<table>
<thead>
<tr>
<th>$Leftmost \ x$</th>
<th>$Rightmost \ x$</th>
<th>$Average \ Rate \ of \ Change$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$3$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$2$</td>
<td></td>
</tr>
<tr>
<td>$0.5$</td>
<td>$1.5$</td>
<td></td>
</tr>
<tr>
<td>$0.75$</td>
<td>$1.25$</td>
<td></td>
</tr>
<tr>
<td>$0.9$</td>
<td>$1.1$</td>
<td></td>
</tr>
<tr>
<td>$0.99$</td>
<td>$1.01$</td>
<td></td>
</tr>
</tbody>
</table>
2. Making your own table, compute $f'(2)$ and $f'(3)$. Do you notice any relationship between $f'(1)$ and these values? Explain why this relationship is occurring.

3. Let $g(x) = x^2 - 4$. Compute $g'(-1)$ and $g'(0)$.

Extension. You saw in the homework that the derivative at a point of a distance-time graph represents the velocity of the object at the given time $t$. (Remember Brett?) Knowing this, what do you think that the derivative at a point of a velocity-time graph would represent? Think about what each secant line represents to help you.

3.17.3 Assessment Questions

Sample assessment questions for lesson 3.17.1.

1. Let $f(x) = \sqrt{x}$. Compute $f'(2)$ by filling out the table below.

<table>
<thead>
<tr>
<th>$Leftmost \ x$</th>
<th>$Rightmost \ x$</th>
<th>$Slope \ of \ the \ Secant \ Line$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>0.99</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>Leftmost $x$</td>
<td>Rightmost $x$</td>
<td>Slope of the Secant Line</td>
</tr>
<tr>
<td>-------------</td>
<td>--------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td>1.99</td>
<td>2.01</td>
<td></td>
</tr>
</tbody>
</table>

2. Suppose that an object is in motion. The *total distance* the object has traveled at any time $t$ is $d(t) = 2t^2$. Find the *velocity* of the object at time $t = 1$ and $t = 2$.

3.18 The Derivative at Local Maxima and Minima

3.18.1 Lesson

This lesson is designed to help the student understand the value of the derivative at a local maximum and local minimum.

1. Let $f(x) = \frac{1}{3}x^3 - \frac{9}{2}x^2 + 18x - 5$. Graph $f$ on the window $[0, 10] \times [-20, 20]$ and sketch the graph below.

2. Trace along the graph of the function on the interval $[2, 4]$. How do the $y$ values of all of the points compare to when $x = 3$? Now trace along the graph of the function on the interval $[5, 7]$. How do the $y$ values of all of the points compare to when $x = 6$?
Because of the properties of the graph of $f$ at $x = 3$ and $x = 6$, the point $(3, \frac{35}{2})$ is called a *Local Maximum* of the function $f$. Likewise, the point $(6, 13)$ is called a *Local Minimum* of the function $f$.

3. In your own words, how would you determine if a point is a local maximum of a function by looking at the graph? How about a local minimum?

4. We now want to find $f'(3)$ and $f'(6)$. Evaluate the average rates of change of at least 5 secant lines (for each) to determine the values of $f'(3)$ and $f'(6)$.

5. What is the relationship between $f'(3)$ and $f'(6)$? Explain.

6. Let’s see if this relationship will occur again. Graph the function $g(x) = \frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x$ on the window $[-5, 5] \times [-10, 10]$ and sketch the graph below.

7. Using the Trace capability, find any local maxima and local minima of $g$. State whether each is a max or min.

8. Evaluate the average rates of change of at least 5 secant lines (for each) to determine the values of $g'$ at the Local Maxima and Minima described in 7.

9. What relationship do you notice about the derivative of $g$ at all of these points? Was this what you expected to happen?

10. Complete the following statement:

    If, for a function $h$, a local maximum or minimum occurs at $(x_0, h(x_0))$, then...

11. Let’s now see if the converse happens. In other words, does the derivative equaling zero imply a local max or min? Let $f(x) = (x - 1)^3 + 1$. Graph this function on the window $[-5, 5] \times [-10, 10]$ and sketch the graph below.
12. Evaluate $f'(1)$ using the average rates of change of at least 5 secant lines.

13. Is there a local max or min at $(1,1)$? Explain.

14. Thus, does the derivative equaling zero imply a local max or min? Explain.

15. List all of the definitions and properties you learned in this lesson.

3.18.2 Homework

Complete this homework after doing lesson 3.18.1.

1. Sketch a graph of the function $f(x) = -(x+2)^2 + 3$. Does this function have a local maximum or minimum? If so, where? Explain.

2. Compute the derivative at the local max or min. What is the value of the derivative at this point?

3. Let $f(x) = (x-1)(x+2)^2$. How many local maxima and local minima will $f$ have? Explain by citing the graph of $f$.

4. Describe at least three situations where the concept of local maxima and local minima may be important.

Extension. Suppose $v(t)$ represents the velocity of an object at any time $t$. If $v'(t_1) = 0$, then explain what the object is doing at time $t_1$. 

93
3.18.3 Assessment Questions

Sample assessment questions for lesson 3.18.1.

1. Sketch the graph of the function \( g(x) = x(x + 1)(x - 4)(x + 2) \). Find the three points where the derivative of \( g \) will be equal to zero. Explain your reasoning.

2. Suppose \( d \) is a function where \( d(t) \) is the distance away from the starting point at any time \( t \). If \( d'(t_1) = 0 \), then explain what the object is doing at time \( t_1 \).

3.19 The Area Under the Curve

3.19.1 Lesson

This lesson is designed to help the student understand the area under a curve.

1. In your calculator, graph \( f(x) = 3x \) on the window \([0, 10] \times [0, 30]\) and sketch the graph below.

2. Recall that when solving inequalities you had to shade in different regions. Similar to what you did in that lesson, shade in the region under the line \( y = 3x \), above the line \( y = 0 \), to the right of the line \( x = 0 \), and the left of the line \( x = 5 \). What type of shape is this region?

3. Find the area of this triangular region.

4. Now find the region under the line \( y = 3x \), above the line \( y = 0 \), to the right of the line \( x = 1 \), and the left of the line \( x = 8 \). What type of shape is this region?
5. Find the area of this region.

6. Now graph the function $g(x) = 5$ on the window $[-5, 5] \times [0, 10]$ and sketch the graph below.

7. Now find the region under the line $y = 5$, above the line $y = 0$, to the right of the line $x = -2$, and the left of the line $x = 4$. What type of shape is this region?

8. Find the area of this region.

Thus far, finding the area under the curve has been not too difficult because all of the regions have made nice geometric shapes. However, as you will see, this is not always the case.

9. Graph the function $h(x) = x^2 + 1$ on the window $[-2, 2] \times [0, 10]$ and sketch the graph below.

Figure 3.9: Sum up the area of these rectangles.
10. Now shade the region under \( y = x^2 + 1 \), above the line \( y = 0 \), to the right of the line \( x = -2 \), and the left of the line \( x = 2 \). Notice that this is not a nice geometric shape of which you know how to find the area. Consider Figure 3.9. Add up the area of the rectangles created (notice that each rectangle has a width of 1 unit and a height equal to the maximum value of the given 1 unit interval.

![Figure 3.10: Sum up the area of these rectangles.](image)

11. Now consider Figure 3.10. Notice that the rectangles now have a width of 0.5 units with a height of the maximum value of the function over the given interval. What is the combined area of the new rectangles? Is this closer to the actual area under the curve than your previous estimate where the rectangles had a width of 1 unit?

12. What do you think will happen to the area of all of the rectangles as the width of the rectangles \( \to 0 \)? Explain your reasoning.
13. Explain in your own words how to compute the area under the curve of the function $f(x) = x^2 + 1$.

Extension. Now complete the method we began in 9 - 12. Make the width of the rectangles $\frac{1}{3}$ and compute the area described by those rectangles. Then make the width of the rectangles $\frac{1}{4}$ and compute the area described by those rectangles. After this, make a conjecture as to what the area under the curve actually is for the function $f(x) = x^2 + 1$ from $x = -2$ to $x = 2$.

3.19.2 Homework

Complete this homework after doing lesson 3.19.1.

1. Compute the area under the curve of $f(x) = 2x + 3$ from $x = -1$ to $x = 5$.

2. Let $f(x) = x^3 + 1$. Approximate the area under the curve from $x = 1$ to $x = 3$. Use rectangles with a width of $\frac{1}{4}$ to approximate.

Extension. Find the area under the curve of $f(x) = x - 2$ from $x = -5$ to $x = 3$. Note that when an area appears below the $x$-axis, then that area is considered “negative area.” Thus, one must add up the area above the $x$-axis and then subtract any area that is below the $x$-axis.

3.19.3 Assessment Questions

Sample assessment questions for lesson 3.19.1.

1. Compute the area under the curve of $f(x) = x + 3$ from $x = -3$ to $x = 5$.

2. Compute the area under the curve of $f(x) = x + 3$ from $x = -1$ to $x = 4$. 

97
3. Approximate the area under the curve of $f(x) = \sqrt{x}$ from $x = 2$ to $x = 4$.

Use rectangles of width $\frac{1}{4}$ to approximate the area.
CHAPTER IV

CONCLUSION

The lessons presented incorporate inquiry and technology in teaching many of the major Calculus concepts such as limits of functions, rates of growth, and tangent lines. They address NCTM's emphasis on inquiry and technology and are intended to be used in grades 8-11 in accordance with the Ohio Department of Education's desires to teach Calculus concepts earlier.

Students using these lessons will develop an understanding of many of the major Calculus concepts along with a better understanding of functions and algebraic notation. Their skills in expressing themselves, solving problems, using technology, and working with others will be strengthened.
BIBLIOGRAPHY


100


